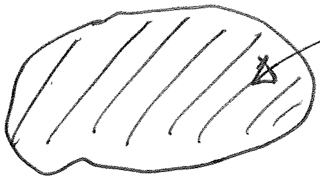
Lecture 4

Open sets (10.1)

<u>Goal</u>: make the notion of 'boundary' precise.



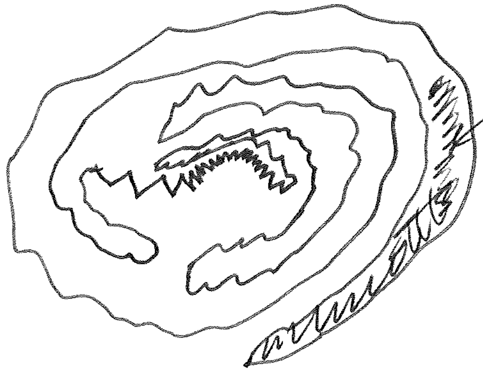
"set of points inside"

C -'the boundary'.

C -curve

"outside"

in this picture, harder to tell "inside" from "outside".



"inside".

(outside the course – Jordan curve theorem)

<u>Need</u>: <u>Good</u> definitions of "inside" and "outside"

let S - set of points (in $\mathbb{R}^2$, or $\mathbb{R}^3$, or in $\mathbb{R}^n$)

<u>will define</u> its <u>interior</u>, <u>exterior</u> and <u>boundary</u> points.

We know what a ball of radius r centered at a point $(a_1, \dots, a_n)$ is:

$$\overline{B}_{a,r} = \{ (x_1, \dots, x_n) \mid (x_1 - a_1)^2 + \dots + (x_n - a_n)^2 \leq r^2 \}$$

<u>Def. of closed ball</u>

<u>closed</u> ball

Def open ball

$$B_{a,r} = \{(x_1, \dots x_n) \mid (x_1-a_1)^2 + \dots + (x_n-a_n)^2 < r\}$$

Def Let S be a set of points in $\mathbb{R}^n$.

A point $a \in S$ is called an **interior** point if

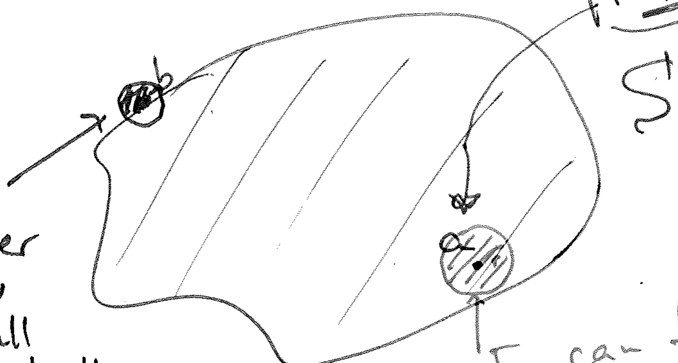
$\exists$ $B_{a,r}$ such that $B_{a,r} \subset S$

$\exists$ (there exists)

$B_{a,r}$: an open ball of radius r centered at a

$B_{a,r} \subset S$: every point of the ball is contained in the set S.

interior point



no matter how small the ball, it has to have some points not in S

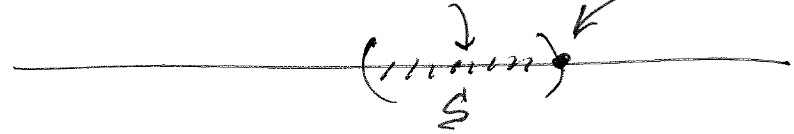
Not an interior point.

I can find r (small) such that ~~the~~ the whole ball is contained in S.

Example on $\mathbb{R}^1$

open ball in $\mathbb{R}^1$.

interior point. boundary point



Def A point b is called a boundary point of S if

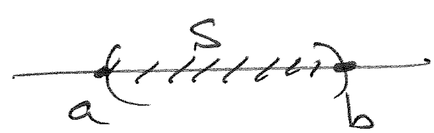
$\forall\ B_{b,r}$, there exists $x \in B_{b,r}$ s.t. $x \notin S$

any ball centered at b

and exists $y \in B_{b,r}$ s.t. $y \in S$.

Examples 1) in $\mathbb{R}^1$

$S = (a,b)$



Boundary of $S = \{a, b\}$.

2 points a, b.

2) Boundary of $\overline{B_{a,r}}$:

sphere of radius r centered at a:

$$S_{a,r} = \{(x_1, \ldots, x_n) \mid (x_1-a_1)^2 + \ldots + (x_n-a_n)^2 = r^2\}.$$

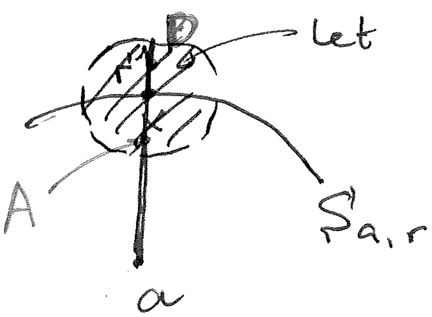
Needs proof: 1) $S_{a,r} \subseteq$ boundary of $\overline{B_{a,r}}$

2) $S_{a,r} \supseteq$ boundary of $\overline{B_{a,r}}$

Proving 1) : Let $x \in S_{a,r}$

Need to show: x is a boundary point of $\overline{B_{a,r}}$.

By definition, this means: need to show that any ball around x contains points in $\overline{B_{a,r}}$ and points that are not in $\overline{B_{a,r}}$.



let C be a ball around x. radius r'.
" $B_{x,r'}$.

Consider the line through a and x.

On this line, there is a point A closer to a than x and a point D farther away from a than x.

The distance from x to a is exactly r.

Then A is in $\overline{B_{a,r}}$
D is outside.

(in fact, could say $x \in \overline{B_{a,r}}$, so ~~more~~ only need D).

2) $S_{a,r} \supseteq$ boundary of $\overline{B_{a,r}}$

Let y be a boundary point of $\overline{B_{a,r}}$.

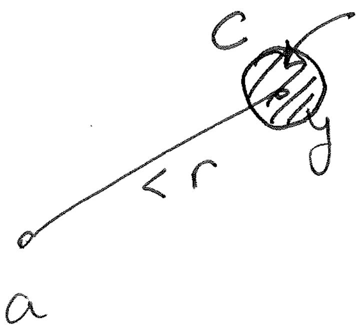
Need to prove: $\underbrace{\text{distance from } y \text{ to } a}_{\text{denote it by } d(y,a)}$ is exactly r.

Proof by contradiction:

suppose it is not.

Then $d(y,a) < r$ or $d(y,a) > r$.

Suppose $d(y,a) < r$.



take a ball $\overset{C}{\vee}$ around y of radius $\dfrac{r - d(a,y)}{2}$.

Let $x \in C$.

Then

$$d(a,x) \le \underline{d(a,y)} + d(y,x)$$

$$< \frac{r - d(a,y)}{2} + d(a,y)$$

$= \dfrac{d(a,y)}{2} + \dfrac{r}{2} < \dfrac{r}{2} + \dfrac{r}{2} = r.$ So, $d(a,x) < r$ then $x \in B_{a,r}$.

We showed that any point $x \in C$ is also contained in $B_{a,r}$.

Then $C \subset B_{a,r}$, then y is an interior point.

Contradiction!

It remains to show that

$d(y,a) > r$ is also impossible.

Similarly, can find a ball around y that is entirely outside $\overline{B_{a,r}}$

(in this case, y is an exterior point to $\overline{B_{a,r}}$)

— again a contradiction.

Exercises: 1) prove that $\overline{B_{a,r}}$ is closed

Def: contains all its boundary points

2) Prove $B_{a,r}$ is open — every point is an interior point.

(generally, strict inequalities define open sets, $\leq$ nonstrict inequalities define closed sets).

think about it

Exercise What is the boundary of a Cantor set?

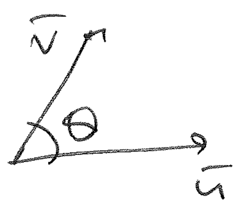
outside this course

Projections (10.2)

Recall: defined dot product,

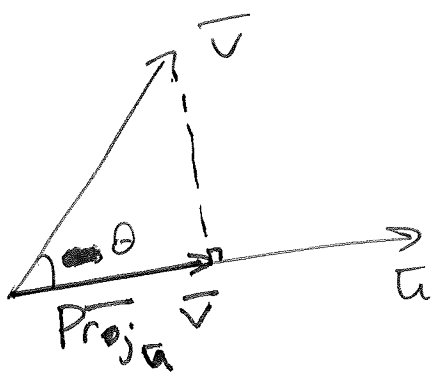
$\bar{u} \cdot \bar{v}$

- $\bar{u} \cdot \bar{v} = 0$ if and only if $\bar{u} \perp \bar{v}$ (perpendicular)
- $\bar{u} \cdot \bar{u} = |\bar{u}|^2$
- $\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cos\theta$



Application: 1) Projection of $\bar{v}$ onto $\bar{u}$ $(\overline{\text{Proj}_{\bar{u}} \bar{v}})$

2) component of $\bar{v}$ along $\bar{u}$
= number
= $\pm$ magnitude of $\overline{\text{Proj}_{\bar{u}} \bar{v}}$



(useful to be able to represent $\bar{v}$ as a sum of a vector parallel to $\bar{u}$ and a vector perpendicular to $\bar{u}$.

The parallel vector is $\text{Proj}_{\bar{u}} \bar{v}$.

How to compute:

1) $\text{comp}_{\bar{u}} \bar{v}$ ← the component

$= \bar{v} \cdot \dfrac{\bar{u}}{|\bar{u}|}$

↑ unit vector in the direction of $\bar{u}$

2) $\text{Proj}_{\bar{u}} \bar{v} = \left(\text{comp}_{\bar{u}} \bar{v}\right) \cdot \dfrac{\bar{u}}{|\bar{u}|}$

$= \dfrac{\bar{v} \cdot \bar{u}}{|\bar{u}|^2} \bar{u}$

Read 10.1 and 10.2 - covered
10.3 and 10.4 - will do next class.
