Reminder:
- Review session Wed Dec 9 3-5 pm in Math 203.
- Office hours:
  - Thurs Dec 10 noon - 2 pm
  - Fri Dec 11 noon - 2 pm
- Will post review materials
- Will have E-S limit.
- Differentiable —
- Also look at Math 200/253 finals.

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Cylindrical and spherical coordinates

Recall: can think about an $SSS$ as
a double integral inside a single integral,
or as a single integral inside a double.

$SSSS (f) dA$ or $\int (SS) dA) dx$

I could convert the double integral to polar coordinates.

Th. $3$ cylindrical coordinates:

$(x, r, \Theta, z)$ or $(x, r, \Theta)$ or $(y, r, \Theta)$

$x$ is itself $y$ (right-hand):

\[ x = r \cos \Theta \]
\[ y = r \sin \Theta \]
\[ z = z \]

$z$ is $z$:

$\Theta$ is $\Theta$:

$\Theta$ is $\Theta$:

(check - is $h_2$ $h_1$ right orientation?)
Example: Find the volume between two paraboloids
\[ z = x^2 + y^2 + 1, \quad z = 2(x^2 + y^2) \]

Method 1: Find the domain \( D \) in the \( xy \)-plane that is the projection of our solid

\[
V = \iiint_D (x^2 + y^2 + 1) \, dA - \iiint_D (2x^2 + 2y^2) \, dA.
\]

We can see that polar coordinates are useful.

Method 2:
\[
V = \iiint_E 1 \, dV = \iiint_D \frac{r^2 + 1}{2r^2} \, dA.
\]

\[
= \int_0^{2\pi} \int_0^1 \int_{r^2}^{r^2 + 1} dz \, r \, dr \, d\theta.
\]

(To find \( \partial D \), we need the radius of the circle of intersection of the paraboloids.)

\[ z = 2x^2 + 2y^2 = x^2 + y^2 + 1 \rightarrow \text{circle of intersection} \]

\[ x^2 + y^2 = 1 \]

So it is a circle of radius 1 (lying in the plane \( z = 2 \)).
Comment: $(x, y, z) \rightarrow (r, \Theta, \phi)$

There is Jacobian formula for integrals in 3D:

so $\, dx\, dy\, dz = \left| \begin{array}{ccc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \Theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \Theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial \phi} \end{array} \right| \, dr\, d\Theta\, d\phi$

abs. value of Jacobian

\begin{bmatrix} \cos \Theta & -r \sin \Theta & 0 \\ \sin \Theta & r \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r.

Spherical coordinates

$f = \sqrt{x^2 + y^2 + z^2} = \text{distance from } (0, 0, 0)$

"rho"

$\phi = \text{"latitude"}$

$0 \leq \phi \leq \pi$

(\text{in geography:} \phi \text{ is measured from the equator,} -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2})

In math, $\phi$ is measured from the North Pole.

On the latitudinal circle, $\Theta$ measures the angle with positive direction of $x$-axis.

$0 \leq \Theta \leq 2\pi$

$x = f \sin \phi \cos \Theta$

$y = f \sin \phi \sin \Theta$

$z = f \cos \phi$

Sanity check: $x^2 + y^2 + z^2 = f^2.$
Exercise: how to express $dV$?

General answer:

\[
\begin{vmatrix}
\frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta}
\end{vmatrix} = \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta
\]

Example: mass of the piece of a sphere above the cone of angular measure $\frac{\pi}{3}$ of radius $5$.

Density function

\[d(x,y,z) = xyz\]

\[
\int_0^{\pi/3} \int_0^{\sqrt{25 - z^2}} \int_0^{\sqrt{25 - z^2 - x^2}} z \, x \, y \, \rho^2 \sin \phi \, d\phi \, d\rho \, dz
\]
Cones in cylindrical coordinates

\[ \tan \psi = \frac{r}{z} \]

Cones in cylindrical coords are given by "linear" equation

\[ r = c z \]

\[ \frac{r}{z} \]

tangent of the angle
defining the cone.