Reminder:

- Review session Wed Dec 9 3-5 pm in Math 203.
- Office hours:

  Thurs Dec 10 noon - 2pm

  Fri Dec 11 noon - 2pm

- will post review materials
  - will have ε-δ limit.
  - differentiable - -
  - also look at math 200 / 253 finals.

Cylindrical and spherical coordinates

Recall: can think about an $\iiint$ as a double integral inside a single integral, a single integral inside a double or as a single

$$\iint \left(\int \quad \right) dA \quad \text{or} \quad \int \left(\iint \_\ \_\ dA \right) dx$$

could convert the double integral to polar coordinates.

This is cylindrical coordinates:

(r, θ, z) or (x, r, θ) or (y, r, θ)

$x = r\cos\theta$
$y = r\sin\theta$

x is itself
$y = r\cos\theta$
$z = r\sin\theta$

y
(x, z) - converted to (r, θ).

$0 \le r \le 1$
$0 \le \theta \le 2\pi$

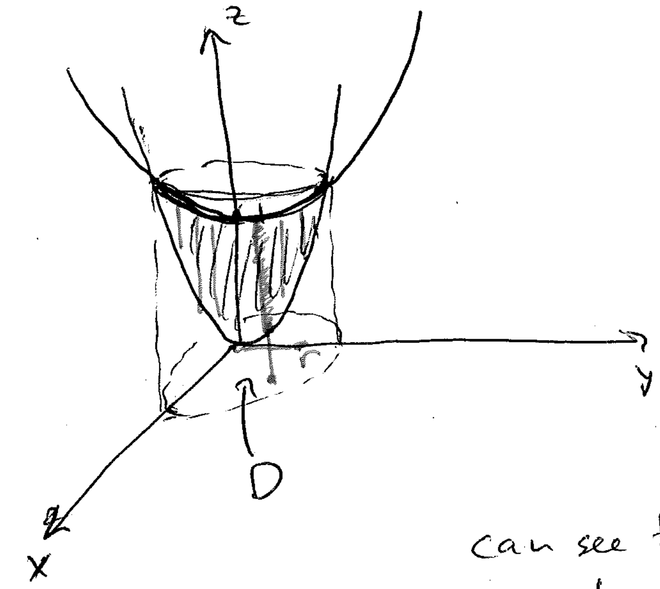
(right-hand:

$\curvearrowleft \theta$

(check - is this the right orientation?)

Example: Find the volume between two paraboloids

$z = x^2+y^2+1$, $z = 2(x^2+y^2)$



can see that polar coords are useful.

Method 1: find the domain D in the xy-plane that is the projection of our solid

$$V = \iint_D (x^2+y^2+1)\,dA - \iint_D (2x^2+2y^2)\,dA.$$

Method 2:

$$V = \iiint_E 1\,dV = \iint_D \int_{2r^2}^{r^2+1} 1\,dz\,dA$$

$$= \int_0^{2\pi}\int_0^{\boxed{1}}\int_{2r^2}^{r^2+1} dz\; r\,dr\,d\theta$$

Jacobian factor from dA.

(To find D, we need the radius of the circle of intersection of the paraboloids.)

$z = 2x^2+2y^2 = x^2+y^2+1$ ← circle of intersection

$\boxed{x^2+y^2=1}$

So it is a circle of radius 1 (lying in the plane $z=2$)

<u>Comment</u>: $(x, y, z) \to (r, \theta, z)$

There is Jacobian formula for integrals in 3d:

so
$$dx\,dy\,dz = \left|\begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{matrix}\right| dr\,d\theta\,dz$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

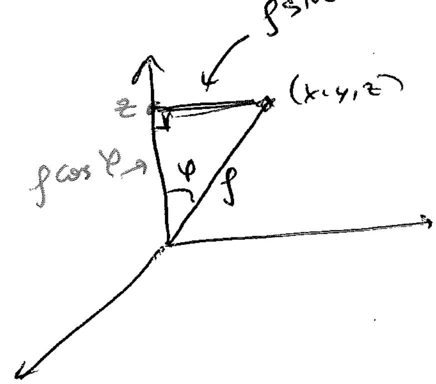
abs. value of Jacobian

$$= \left|\begin{matrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{matrix}\right| = r.$$

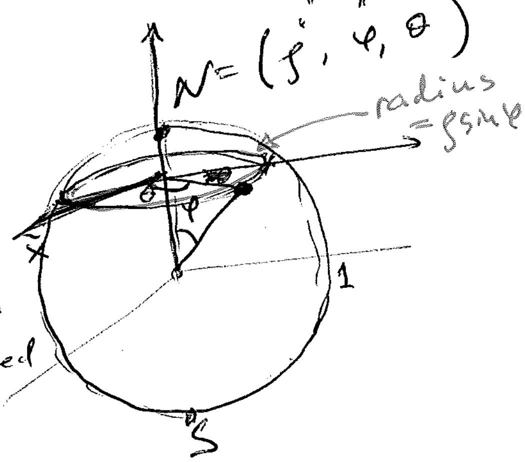
Spherical coordinates

$\rho = \sqrt{x^2+y^2+z^2}$ = distance from $(0,0,0)$

"rho"



$N = (\rho, \varphi, \theta)$ — $\rho = 1$, $\varphi = 0$, θ undefined.



φ = "latitude"

$0 \le \varphi < \pi$

(in geography: φ is measured from the equator, $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$)

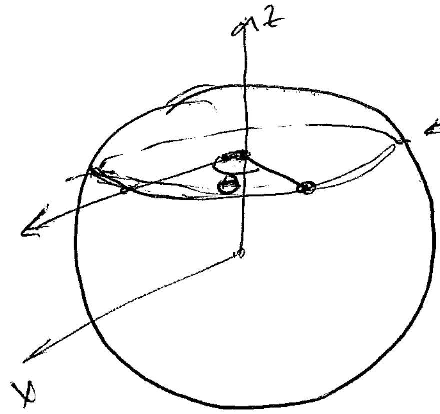
In math, φ is measured from the North Pole.

On the latitudal circle, θ measures the angle with positive direction of x-axis.

$0 \le \theta < 2\pi$

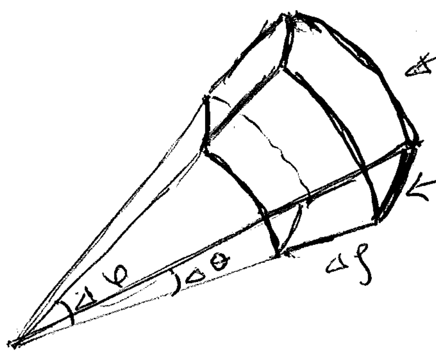
$$\boxed{\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}}$$

Sanity check: $x^2+y^2+z^2 = \rho^2$.

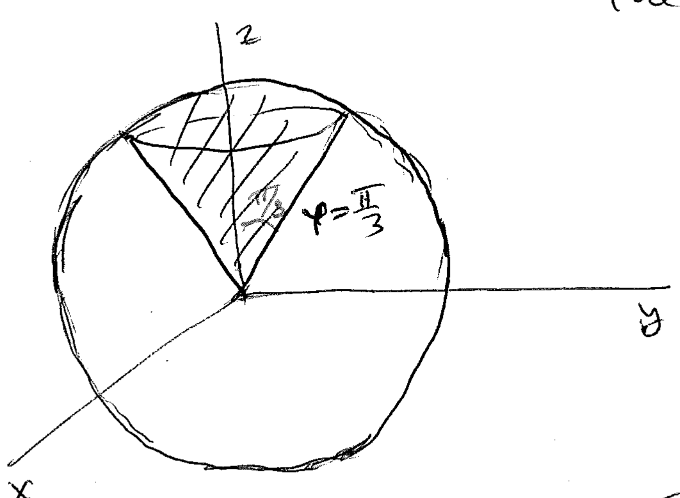


Exercise: how to express dV?

General answer:
$$\left|\begin{matrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{matrix}\right| = \underline{\rho^2 \sin\varphi}\, d\rho\, d\varphi\, d\theta$$



Example mass ~~volume~~ of the piece of a sphere of radius 5 above the cone of angular measure $\frac{\pi}{3}$



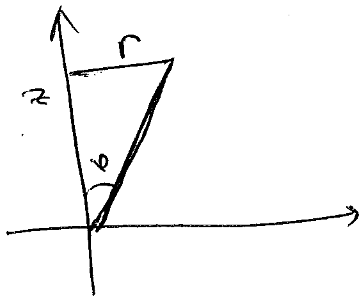
density function
$$d(x,y,z) = xyz.$$

$$\int_0^5 \int_0^{\pi/3} \int_0^{2\pi} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\boxed{\rho \sin\varphi \cos\theta \cdot \rho \sin\varphi \sin\theta \cdot \rho \cos\varphi}$$

was $d(x,y,z)$

Cones in cylindrical coordinates



$$\tan\varphi = \frac{r}{z}.$$

Cones in cylindrical coords are given by "linear" equation

$$r = cz$$

tangent of the angle defining the cone.