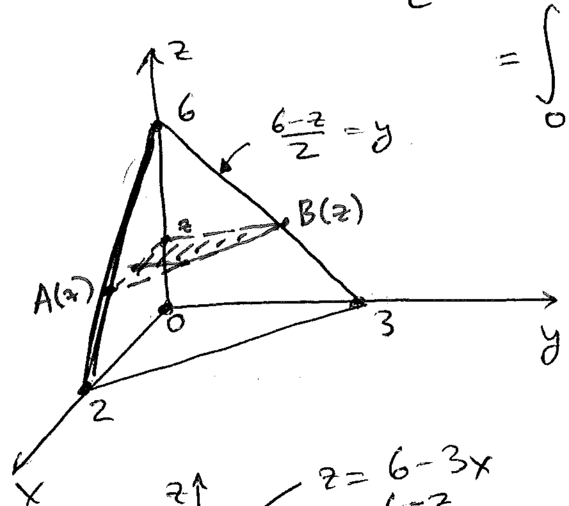
Iterated integrals in $\mathbb{R}^3$

<u>Example</u> Compute total mass of the solid bounded by the coordinate planes and the plane $z = 6 - 3x - 2y$ with density function $\rho(x,y,z) = xy + z$.

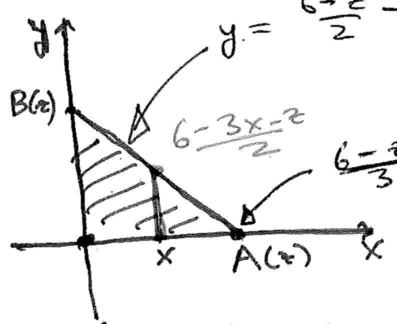
<u>Solution</u>: E - our solid

$$M = \iiint_E \rho(x,y,z)\, dV.$$

$$= \underbrace{\int_0^6 \int_0^{\frac{6-z}{3}} \int_0^{\frac{6-3x-z}{2}} xy + z \; dy\, dx}_{\text{integral over}}\, dz =$$



$$y = \frac{6-z}{2} - \frac{\frac{6-z}{2}}{\frac{6-z}{3}}\, x$$

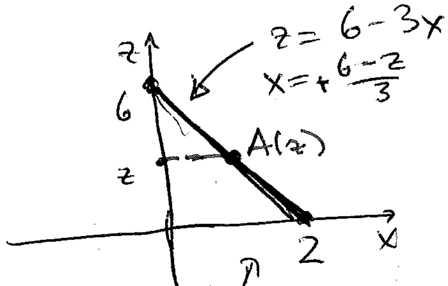


$$= \frac{6-z}{2} - \frac{3}{2}x$$

$$= \frac{6-3x-z}{2}$$

eq. of the plane.

Need to find $A(z)$, $B(z)$.



this line is the intersection of the xz-plane with $z = 6 - 3x - 2y$.

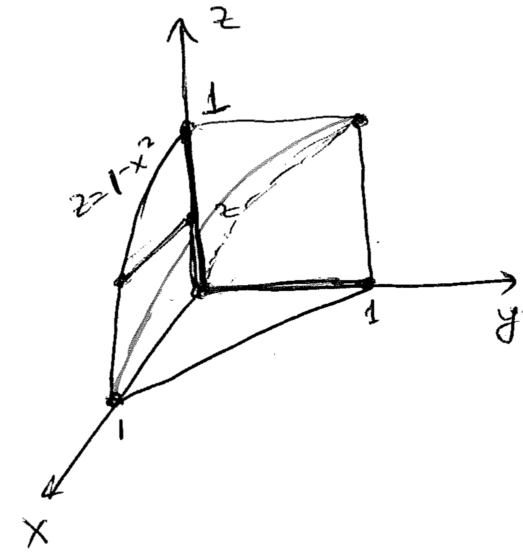
$$= \int_0^6 \int_0^{\frac{6-z}{3}} \underbrace{\left(x\frac{y^2}{2} + zy\right)\Big|_{y=0}^{\frac{6-3x-z}{2}}}_{\text{fn of } (x,z).} dx\, dz$$

$= \ldots$

Example:

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z) \quad dy\,dz\,dx$$

Rewrite it in the order $\iiint dx\,dy\,dz$



$0 \le x \le 1$

$0 \le z \le 1-x^2$

$0 \le y \le 1-x$

$$= \int_0^1 \int_0^1 \qquad dy\,dz$$

Need to split according to the constraint on x.

limits for z have to be numbers.

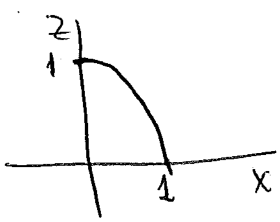
Min possible z is 0.

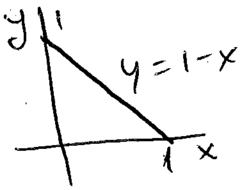
Max possible z:

$z \le 1-x^2$ ← max when x is min.

Min $x = 0$.

So max $z = 1$.





Now need inequalities for x, y for any given fixed z.

$z \le 1-x^2$
$$\boxed{\begin{array}{l} x^2 \le 1-z \\ 0 \le x \le \sqrt{1-z} \end{array}}$$

$y \le 1-x$. Looking for max possible y (because y is next in our integral)

$$\iiint \boxed{dy}\,dz$$

y is max when x is min. Even with z fixed, x is min when $x=0$.

So max $y = 1$.

Limits for x:

Now (y,z) are fixed.
We have to have:

$$x^2 \le 1-z, \qquad 0 \le x \le \sqrt{1-z}.$$

also, we had $y \le 1-x$. so $x \le 1-y$.

We have 'competing' estimates,

$$x \le \sqrt{1-z} \quad \underline{\text{and}} \quad x \le 1-y.$$

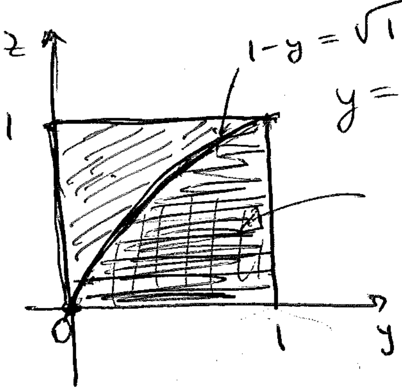
At this point y, z are both fixed.
Which one is more restrictive?

$$\sqrt{1-z} \ \underset{?}{\gtrless}\ 1-y$$

$$1-y = \sqrt{1-z}$$

$$(1-y)^2 = 1-z$$

$$z = 1-(1-y)^2$$



So, when $y > 1-\sqrt{1-z}$, we have that $1-y < \sqrt{1-z}$,
so the limit for x is $1-y$.

when $0<y < 1-\sqrt{1-z}$, then $\sqrt{1-z} < 1-y$, so
the limit for x is $\sqrt{1-z}$.

We get:

$$\underbrace{\int_0^1\int_0^{1-\sqrt{1-z}}}_{\substack{\text{integral over}\\ \text{the green region}\\ \text{in the } y\text{-}z \text{ plane}}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz$$

$$+ \underbrace{\int_0^1\int_{1-\sqrt{1-z}}^1}_{\substack{\text{integral over the black region}\\ \text{in the } yz\text{-plane}}} \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz.$$