Change of variables in double integrals

Last time: $x = x(u,v)$
$y = y(u,v)$

$$\iint_D f(x,y)\,\underbrace{dx\,dy}_{dA} = \iint_{D'} f(x(u,v),y(u,v)) \overbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}^{\text{Jacobian}} du\,dv$$

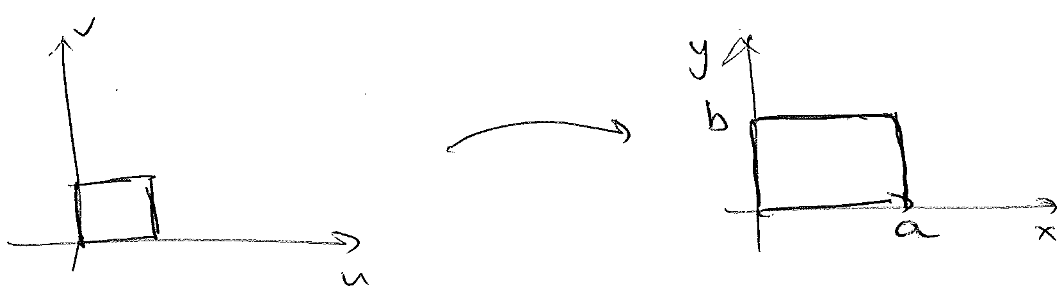
D': same domain expressed using (u,v)-coordinates

(example: polar change)
$x = r\cos\theta$
$y = r\sin\theta$

Proof: imagine we did just a simple linear change of variables: $x = au$, $y = bv$

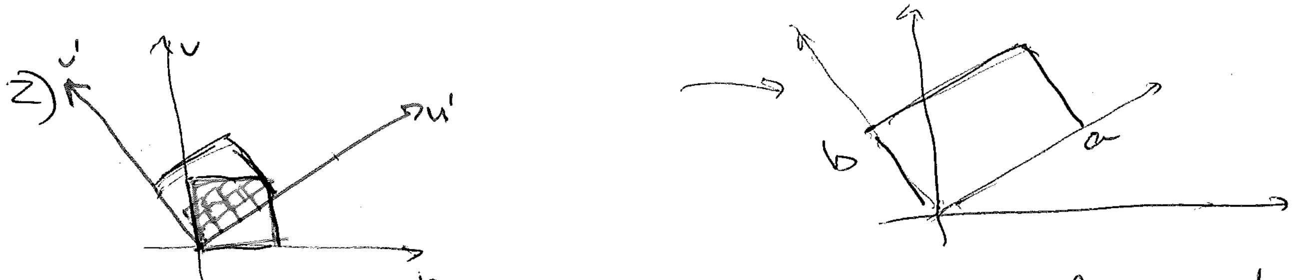
$dx = a\,du$
$dy = b\,dv$
$dx\,dy = ab\,du\,dv$

1)



The area of each rectangle gets multiplied by ab.

2)



suppose (u',v') - other coordinates, and our linear transf. is a scaling by a along u', b along v'.

This linear transformation also changes the areas by the factor ab.

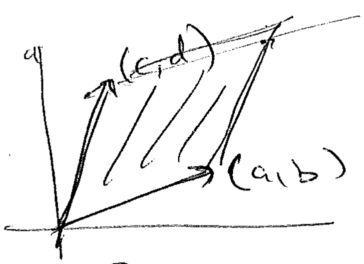
• You can diagonalize most linear transformations by a change of basis.
For those where you cannot, it does not matter for the areas.

the determinant does not change.

Summary: A linear transformation multiplies areas by its det.:

let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Then if R is a rectangle, $T(R)$ is a parallelogram of area $\det(T) \cdot \text{area}(R)$.

Note: the book has a proof using the formula for the area of a parallelogram using cross product.



Area $= \begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

image of  under the transformation with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

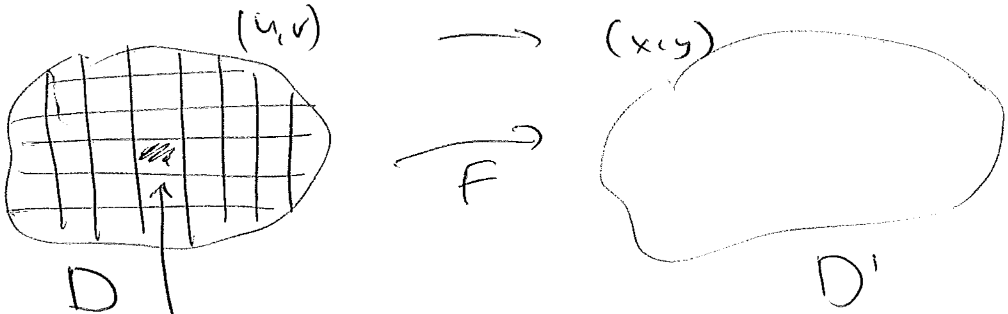
• Our proof works in any $\mathbb{R}^n$.

(!) $T : \mathbb{R}^n \to \mathbb{R}^n$ - linear
volumes get multiplied by $\det(T)$.

- Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a ~~linear~~ differentiable transformation.

$$\underset{u,v}{\mathbb{R}^2} \longrightarrow \underset{x,y}{\mathbb{R}^2}$$

$$F(u,v) = \big(\underline{x(u,v)},\ \underline{y(u,v)}\big).$$



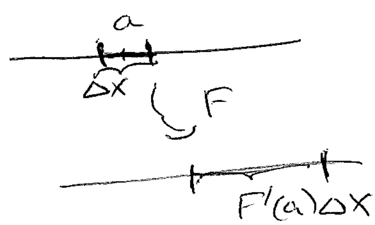
replace F with its linearization.

The ~~matrix of the linearization is:~~ the Jacobian $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ $\left|\ \cdot\ \right|$

The claim is that the area of the image of a small rectangle under F is $\underline{\text{very close}}$ to the area of the image of that rectangle under the linearization of F.

Then $$\underbrace{dx\,dy}_{\text{area of the image of }▨} \approx \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| \underbrace{du\,dv}_{\text{area of }▨}.$$

<u>Note:</u> in one dimension, have seen this already.



length of the image of a small interval around a is approximately $F'(a)\Delta x$

• Points of caution:

1) To use the Jacobian formula, the transformation F needs to be one-to-one on the domain D.

(one-to-one means: if $(x_1, y_1) \neq (x_2, y_2)$ and both in D,

then $F(x_1, y_1) \neq F(x_2, y_2)$.)

If not, subdivide D.

2) I talked about the change $x = x(u,v)$, $y = y(u,v)$

Often it is convenient to have $u = u(x,y)$, $v = v(x,y)$.

Then need the Jacobian
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}^{-1}.$$

(Recall Inverse Function Thm).

The Jacobian formula works in any $\mathbb{R}^n$.

Triple integrals

Def: E - solid in $\mathbb{R}^3$.

$$\iiint_E f(x,y,z)\,dV = \text{limit of Riemann sums:}$$

subdivide E into small cubes
pick a point from each, evaluate $f(x^*, y^*, z^*)$, make a Riemann sum.

Converges for continuous f.

Main skill: writing such integrals as iterated integrals.

Example: $E = \overset{x}{[a,b]} \times \overset{y}{[c,d]} \times \overset{z}{[e,f]} = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{array} \right\}$

$$\iiint_E f(x,y,z)\,dV = \underline{\int_a^b} \underbrace{\int_c^d \underbrace{\int_e^f f(x,y,z)\,dz}_{\text{fn of } x,y}\,dy}_{\text{fn of } x}\,dx$$

When E is not a box, limits are functions describing the shape of E.

Careful: outside integral should have const. limits

$$\int_{\underline{\underline{a}}}^{b} \underline{\qquad\qquad}\, dx \;;$$

can use x for limits $\underline{\underline{\text{inside}}}$ $\int dx$.