Goal: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx \stackrel{?}{=} \sqrt{\pi}$

Will use 2-variable techniques.

- What do we mean by improper integrals in 2 variables.

Thm (Fubini) For nonnegative functions.

$f(x,y) \geq 0$ on a domain D

(the integral may be improper: D may not be bounded,

or $f(x,y)$ might have a discontinuity,

maybe $f(x,y) \underset{(x,y)\to(a,b)}{\longrightarrow} \infty$)

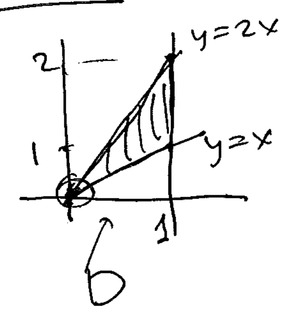
then $\displaystyle\iint_D f(x,y)\,dA$ converges (enough for one of them to converge, then all converge)

iff $\displaystyle\iint f(x,y)\,dx\,dy$ and $\displaystyle\iint f(x,y)\,dy\,dx$

converge.

In practice, for non-negative functions, we will replace the double integral with an iterated integral, and then deal with single-variable improper integrals.

Examples 1) $\displaystyle\iint_D \frac{1}{x\sqrt{y}}\,dA$



Note: improper b/c $\dfrac{1}{x\sqrt{y}} \to \infty$ as $(x,y)\to(0,0)$

1) $f(x,y) \geq 0$ so replace it with iterated integral.

$$= \int_0^1 \int_x^{2x} \frac{1}{x\sqrt{y}}\, dy\, dx = \int_0^1 \left(\frac{1}{x} \cdot 2y^{1/2} \Big|_x^{2x} \right) dx$$

$$= 2\int_0^1 \frac{1}{x} \cdot (\sqrt{2}-1)\sqrt{x}\, dx = 2(\sqrt{2}-1) \underbrace{\int_0^1 \frac{1}{\sqrt{x}}\, dx}_{\text{improper}}$$

$$= 2x^{1/2} \Big|_0^1$$

(Recall: $\lim_{T \to 0} \int_T^1 \frac{1}{\sqrt{x}}\, dx = 2 x^{1/2} \Big|_T^1$)

$$= 4(\sqrt{2}-1)$$

What if $f(x,y)$ is NOT nonnegative:

- Consider $|f(x,y)|$.

If for $|f(x,y)|$ $\iint |f(x,y)|\, dx\, dy$

or $\iint |f(x,y)|\, dy\, dx$

or $\iint_D |f(x,y)|\, dA$ converges,

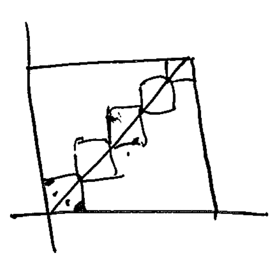
then again you can replace $\iint_D f(x,y)\, dA$

with an iterated integral in any order.

Example (caution!) (homework)

$$\iint_D \frac{x-y}{(x+y)^3}\, dA$$

D: $0 < x < 1$
$0 < y < 1$.



Both iterated integrals converge.

But the double integral does not converge!!

In this example,

$$\iint_D \left| \frac{x-y}{(x+y)^3} \right| dA \quad \text{is infinite.}$$

<u>Strategy</u>: When you see an integral (even over a closed bounded domain)

- check if it is improper (look for discontinuities of $f(x,y)$)

- if it is improper, see if $f(x,y) \geq 0$ on D.

  If yes, replace with an iterated integral and proceed (will get some improper single-variable integral)

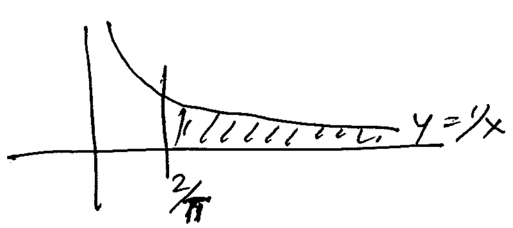
- if NOT nonnegative, consider

  $\iint_D |f(x,y)|\, dA$; replace that with iterated integral, see if it converges.

  <u>If</u> yes, then can replace $\iint_D f(x,y)\, dA$ with iterated integrals, compute.

  If NO, then stop.

<u>Example</u>: $\displaystyle\iint_R \frac{1}{x} \sin\frac{1}{x}\, dA$ $\qquad \frac{2}{\pi} \leq x < \infty$, $\quad 0 \leq y \leq 1/x$



when $x > 2/\pi$

$0 < \frac{1}{x} < \frac{\pi}{2}$

So $\sin\frac{1}{x} > 0$.

So our function is positive, can use iterated integral.

$$\iint_R \frac{1}{x}\sin\frac{1}{x}\,dA = \int_{2/\pi}^{\infty}\int_0^{1/x} \frac{1}{x}\sin\frac{1}{x}\,dy\,dx$$

$$= \int_{2/\pi}^{\infty} \frac{1}{x^2}\sin\frac{1}{x}\,dx \underset{\substack{u=\frac{1}{x}\\ du=-\frac{1}{x^2}dx}}{=} -\int_{\pi/2}^{0} \sin u\,du = 1.$$

– converges.

Also, when the improper integral converges <u>absolutely</u>, can do any change of coordinates (e.g. polar).

<u>Example</u> Our $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

<u>Trick</u>: square it,

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} \underbrace{e^{-r^2}\cdot r\,dr}\,d\theta$$

function is positive so can change to polar.

$u = r^2$

$du = 2r\,dr$

$$\boxed{\int_a^b \int_c^d f(x)g(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)}$$

$x^2+y^2=r^2$

$$= 2\pi\cdot\frac{1}{2}\underbrace{\int_0^{\infty} e^{-u}\,du}_{} = \pi.$$

$$\lim_{T\to\infty} -e^{-u}\Big|_0^T = -e^{-u}\Big|_0^{\infty} = e^0 = 1.$$

Preview: change of variables formula.

$$\iint_D f(x,y)\, dA \rightsquigarrow \iint_{D'} f(u,v)\, du\, dv.$$

$x = x(u,v)$

$y = y(u,v)$

D' ↑ domain in (u,v)-coordinates

Need a relation between $dx\,dy$ and $du\,dv$.

Make a Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$dx\,dy = \left|\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}\right| du\,dv.$$

fn of (u,v)

Example: $x = r\cos\theta$ $\quad dx\,dy = \boxed{r}\,dr\,d\theta$

$y = r\sin\theta$

Jacobian: $\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r.}$