Today: improper integrals (14.3)
Mean Value Thm
More examples from 14.4.

D is closed and bounded unless otherwise specified.

Towards Mean Value Thm:

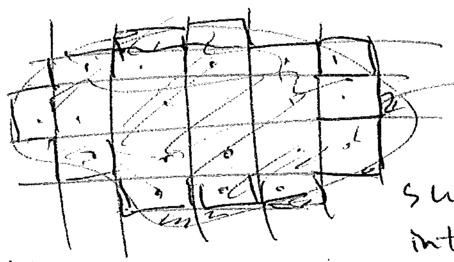
Recall: $\text{Area}(D) = \iint_D 1\, dA$.

Def: Average value of $f(x,y)$ over a domain D

is $\frac{1}{\text{area}(D)} \iint_D f(x,y)\, dA$.

(Why it makes sense: average of f over the points $(x_1,y_1), \ldots, (x_n,y_n)$

should be $\frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i)$.



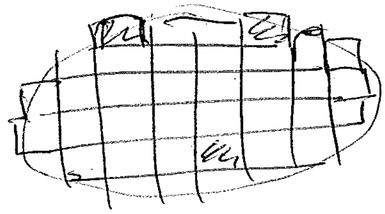
subdivide D into equal rectangles say of area ΔA

the number of rectangles.

$$\underbrace{\frac{1}{n^2 \Delta A} \sum_{\substack{i=1\ldots n \\ j=1\ldots n}} \Delta A\, f(x_i,y_j)}_{\text{average of } f(x_i,y_j)} = \frac{1}{n^2 \Delta A} \cdot \left(\text{Riemann sum of the } \iint_D f(x,y)\, dA\right)$$

Claim: $\text{Area}(D) = \lim_{n\to\infty} (n^2 \Delta A)$.)

Again: why $\text{Area}(D) = \iint_D 1 \, dA$.



$$\text{Area} \approx \text{(area of a rectangle } \square\text{)} \cdot \underbrace{\text{their number}}$$

to get an overestimate, count all that have a point in D.

To get an underestimate, count only the ones entirely in D.

as the size of $\square$ goes to 0, the difference goes to 0.
(if the boundary of D is nice enough: smooth is sufficient).

both are Riemann sums of the function $f(x,y)$ defined by:

$\boxed{\text{characteristic function of } D.}$

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

Estimating integrals:

$m \le f(x,y) \le M$ for $(x,y) \in D$.

Then

$$\boxed{m\,\mathrm{Area}(D) \le \iint_D f(x,y)\,dA \le M \cdot \mathrm{Area}(D).}$$

(Generally, if $f(x,y) \le g(x,y)$ on D

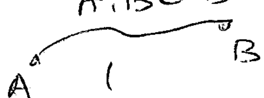
then $\displaystyle\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$)

because this is true for Riemann sums.

Mean Value Theorem for integrals

D – closed, (path) connected, bounded domain; f continuous on D.

$A, B \in D$.



can get from A to B inside D

Then exists $(x_0, y_0) \in D$ s.t.

$$f(x_0,y_0) = \text{average value of } f \text{ over } D = \overbrace{\frac{1}{\mathrm{Area}(D)}\iint_D f(x,y)\,dA}^{A(f)}$$

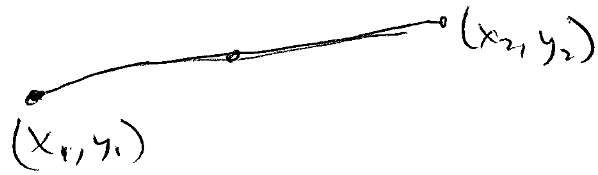
Proof: Let $m = \min_D f(x,y)$, $M = \max_D f(x,y)$.

(both exist b/c D is closed, bounded, f is continuous).

Then $\quad m \le A(f) \le M.$

Let (x_1, y_1) be the point where f attains its min value

(x_2, y_2) — , — its max.



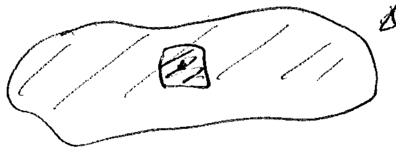
then there is a path in D connecting (x_1, y_1) and (x_2, y_2).

$f(x,y)$ is continuous, so by intermediate value Thm, there is (x_0, y_0) on this path where

$$f(x_0, y_0) = A(f).$$

Corollary: Every object has a centre of mass!

Imagine a metal plate, of variable density.



$$\rho(x,y) = \text{density}.$$

$$= \lim_{\text{size}(\square) \to 0} \frac{\text{mass of } (\square)}{\text{area of } (\square)}$$

$$\text{Total mass} = \iint_D \rho(x,y)\, dA$$

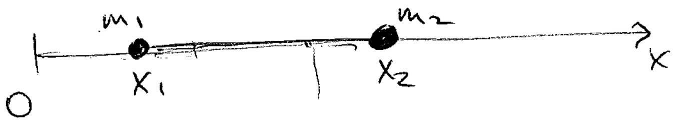
Coordinates of the centre of mass are $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{1}{\cancel{\text{Area}}\ \text{Mass}(D)} \iint_D x\, \rho(x,y)\, dA$$

$$\bar{y} = \frac{1}{\cancel{\text{Area}}\ \text{Mass}(D)} \iint_D y\, \rho(x,y)\, dA$$

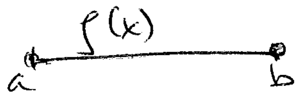
$$\text{Mass}(D) = \iint_D \rho(x,y)\, dA.$$

Why should these formulas make sense:



$$\text{centre of mass} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

= "weighted average" of x_1 and x_2.



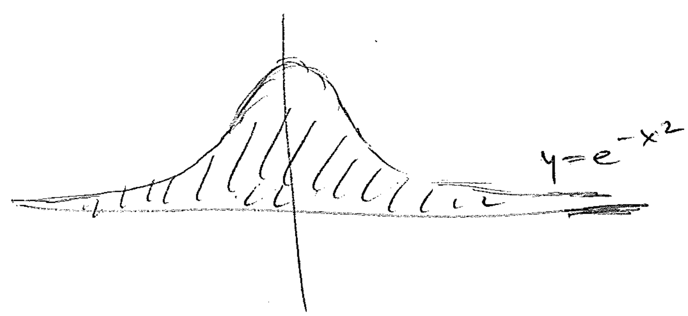
$$\text{centre of mass} = \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$

intuition: mass of a small interval around x $\approx$ "$\rho(x)\underset{\Delta x}{dx}$"

Our formulas are 2-variable version of this.

Next: improper integrals.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$



next time.