Today: improper integrals
Mean Value Thm (14.3)
More examples from 14.4.

Towards Mean Value Thm:

Recall: Area(D) = \( \iint_D 1 \, dA \).

Def: Average value of \( f(x,y) \) over a domain \( D \)
\[ \frac{1}{\text{area}(D)} \iint_D f(x,y) \, dA. \]

(Why it makes sense: average of \( f \) over two
points \( (x_1, y_1), \ldots, (x_n, y_n) \)
should be \( \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i) \).

Subdivide \( D \) into equal rectangles
say of area \( \Delta A \)

\[
\frac{1}{n^2 \Delta A} \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta A \cdot f(x_i, y_j) = \frac{1}{n^2 \Delta A} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j) \quad \text{(Riemann sum of the}
\iint_D f(x,y) \, dA).
\]

Claim: \( \text{Area}(D) = \lim_{n \to \infty} (n^2 \Delta A) \).
Again: why $\text{Area}(D) = \int \int_\text{d}A$.

Area $\approx \text{(area of a rectangle)} \cdot \text{their number}$.

To get an overestimate, count all that have a point in $D$.

To get an underestimate, count only the ones entirely in $D$.

As the size of $\vp$ goes to 0, the difference goes to 0.

If the boundary of $D$ is nice enough: smooth is sufficient.

Characteristic function of $D$.

\[ f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases} \]
Estimating integrals:

\[ m \leq f(x,y) \leq M \quad \forall \quad (x,y) \in D. \]

Then

\[ m \cdot \text{Area}(D) \leq \iint_D f(x,y) \, dA \leq M \cdot \text{Area}(D). \]

(Generally, if \( f(x,y) \leq g(x,y) \) on \( D \)

\[ \iint_D f(x,y) \, dA \leq \iint_D g(x,y) \, dA \]

because this is true for Riemann sums.

Mean Value Theorem for Integrals

\( D \) - closed, connected, bounded domain; \( f \) continuous on \( D \).

\[ A, \ B \in D. \]

\[ A \rightarrow B \]

can get from \( A \) to \( B \) inside \( D \)

Then there exists \((x_0,y_0) \in D\) such that

\[ f(x_0,y_0) = \text{average value of } f \text{ over } D = \frac{1}{\text{Area}(D)} \iint_D f(x,y) \, dA. \]

Proof: Let \( m = \min_{(x,y) \in D} f(x,y) \), \( M = \max_{(x,y) \in D} f(x,y) \).

(both exist \( \because D \) is closed, bounded, \( f \) is continuous).
Then $m \leq A(f) \leq M$.

Let $(x_1, y_1)$ be the point where $f$ attains its minimum value.

$(x_2, y_2)$ is the point where $f$ attains its maximum value.

Then there is a path in $D$ connecting $(x_1, y_1)$ and $(x_2, y_2)$.

Let $f(x, y)$ be continuous, so by intermediate value theorem, there is $(x_0, y_0)$ on this path where $f(x_0, y_0) = A(f)$.

**Corollary:** Every object has a center of mass!

Imagine a metal plate of variable density.

![A metal plate with density function $f(x, y)$ and mass, area, and density concepts illustrated.]

Total mass $= \iint_D f(x, y) \, dA$

The coordinates of the center of mass are $(\bar{x}, \bar{y})$:

\[
\bar{x} = \frac{1}{\text{mass}(D)} \iint_D x \, f(x, y) \, dA
\]

\[
\bar{y} = \frac{1}{\text{mass}(D)} \iint_D y \, f(x, y) \, dA
\]

\[
\text{mass}(D) = \iint_D f(x, y) \, dA
\]
Why should these formulas make sense:

centre of mass = \( \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} \)

= "weighted average" of \( x_1 \) and \( x_2 \).

\[
\text{centre of mass} = \frac{\int_a^b x f(x) \, dx}{\int_a^b f(x) \, dx}
\]

intuition: mass of a small interval around \( x \) \( \approx \frac{f(x) \, dx}{dx} \)

Our formulas are 2-variable version of this.

Next: improper integrals.

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx
\]

next time.