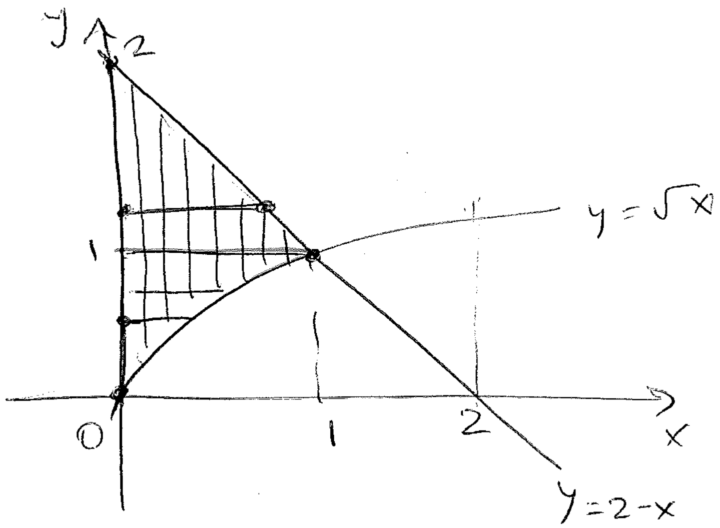
Example: Switch the order of integration in

$$\int_0^1 \int_{\sqrt{x}}^{2-x} f(x,y)\,dy\,dx = \underbrace{\int_0^2 \int_0^{??} f(x,y)\,dx\,dy}_{\text{cannot write a single integral.}}$$

breaks into a sum of two:




to find the limits, express the boundaries as $x = f(y)$, $x = g(y)$.

$$y = \sqrt{x} \iff x = y^2$$
$$y = 2-x \iff x = 2-y$$

When $0 \le y \le 1$, then $0 \le x < y^2$

$1 \le y \le 2$, then $0 \le x \le 2-y$

Answer: $$\int_0^1 \int_0^{y^2} f(x,y)\,dx\,dy + \int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy$$

Remark: without a picture, could do it by looking at inequalities:

$$\begin{cases} 0 \le x \le 1 \\ \sqrt{x} \le y \le 2-x \end{cases}$$ — try to rewrite the inequalities so that $f(y) \le x \le \underline{\underline{g(y)}}$.

First step: For $y$ (because it is on the outside integral), we have to find max possible and min possible value.

~~the~~ For min $y$:

given: $y \geq \sqrt{x}$.
↑ what is given about $x$?
$x \geq 0$

So, $y \geq 0$.

max $y$: $y \leq 2-x$, and $0 \leq x \leq 1$.

$y$ is biggest when $x$ is the smallest.

so $y \leq 2-0=2$.

Get: $0 \leq y \leq 2$.

Now try to find the limits for $x$:

$\sqrt{x} \leq y \leq 2-x$, and $0 \leq x \leq 1$.

$\sqrt{x} \leq y \iff x \leq y^2$ (since $x, y \geq 0$).

$y \leq 2-x \iff x \leq 2-y$.

'both have to hold.

"competing inequalities". We get: $x \leq \min(y^2, 2-y)$

Then analyze which one is "stricter":

compare $y^2$ and $2-y$.

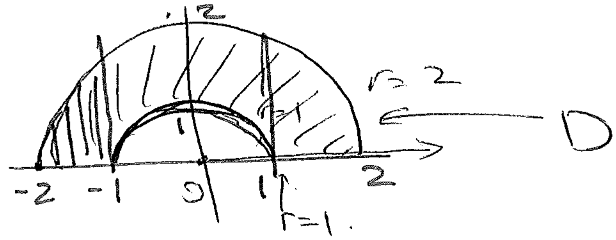
... (decide when $y^2-(2-y) \geq 0$?)

Get: when $0 \leq y \leq 1$, then $y^2$ is smaller, so it "wins".

when $1 \leq y \leq 2$, $2-y$ wins. Same answer.

# Polar coordinates :

Want to integrate $f(x,y)$ over $D$




$$\int_?^? \int_?^? \; dx\,dy \quad \text{or} \quad dy\,dx \quad \text{is not nice.}$$

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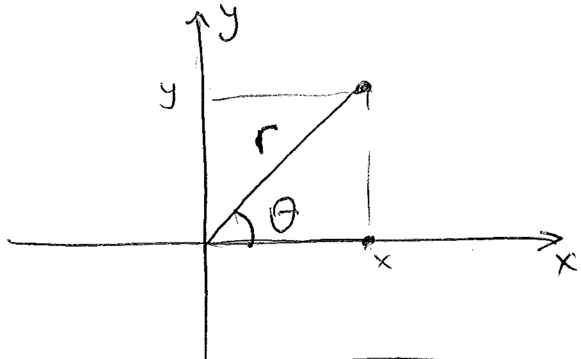



Instead of $(x,y)$, can use $(r,\theta)$ to describe a point on the plane.

$r \geq 0$

$0 \leq \theta < 2\pi$

For $(0,0)$ : $r=0$, $\theta$ doesn't matter.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r^2 &= x^2+y^2 \\ \tan\theta &= y/x \end{aligned}}$$

Our domain $D$ between the two circles in the upper half-plane now has a nice description:

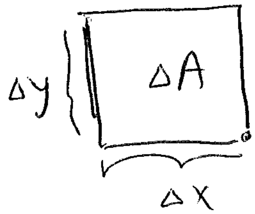
$$\boxed{\begin{aligned} 1 &\leq r \leq 2 \\ 0 &\leq \theta \leq \pi \end{aligned}}$$

"rectangle" in polar coordinates

Setting up an integral over this domain:

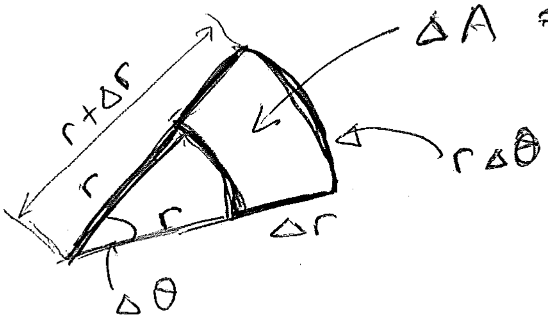
$$\iint_D f(x,y)\,dA = \int_1^2 \int_0^\pi \underbrace{\boxed{?}}_{f(r\cos\theta,\; r\sin\theta)}\, d\theta\, dr$$

what happened to "$dA$"




$\Delta A = \Delta x \Delta y$.

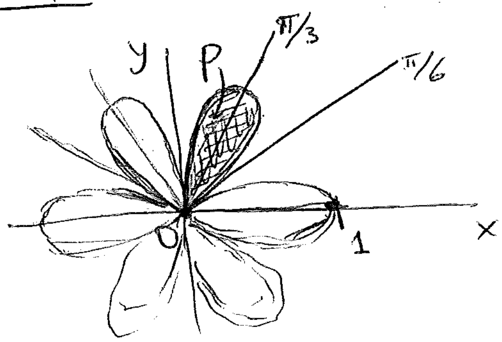
"$dA = dx\,dy$"




$\Delta A \approx r\,\Delta\theta\,\Delta r$

$$\iint_D f(x,y)\,dA = \int_1^2\int_0^{\pi} f(r\cos\theta, r\sin\theta)\cdot r\,d\theta\,dr$$

Example: $r = \cos^2 3\theta$   Find area(P).




$$A(P) = \iint_P 1\,dA$$

$$= \int_{\pi/6}^{\pi/2}\int_0^{\cos^2 3\theta} 1\cdot\boxed{r}\,dr\,d\theta$$

came from the change of variables.

$$= \int_{\pi/6}^{\pi/2} r^2/2\Big|_0^{\cos^2 3\theta}\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/2}\cos^4 3\theta\,d\theta = \frac{1}{6}\int_{\pi/2}^{3\pi/2}\cos^4 u\,du$$

$u = 3\theta$

$$\boxed{\cos^2 u = \frac{\cos 2u + 1}{2} \quad \text{use it twice}}$$

* (or wolframalpha once ☺)

$$= \ldots \frac{1}{6}\cdot\frac{1}{32}\left(12u + 8\sin(2u) + \sin(4u)\right)\Big|_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{16}\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) + 0 = \boxed{\frac{\pi}{16}}$$