Last time: defined integral in $\mathbb{R}^2$ with resp. to area.
(Riemann sums).

Always assume: $f(x,y,z)$ is continuous
(piecewise-continuous at worst).

Domain is bounded by a simple curve
no self-intersections

If we have a region like this, will break it into
a union of simpler regions.

Key point of today:
To evaluate a double integral, you need to
represent it as an iterated integral.

Ex:

$$R = [a,b] \times [c,d]$$

Domain is a rectangle.
Claim: \[ \iint_R f(x,y) \, dA = \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx \]

\[ = \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy \]

for every fixed \( y \), compute \( \int_a^b f(x,y) \, dx \) as the usual way (keep \( y \) as a parameter).

A case of Fubini's Theorem.

Example: \( R = [1,2] \times [3,4] \)

Evaluate \( \iint_R (x + \cos y) \, dA \)

\[ = \int_1^2 \left( \int_3^4 (x + \cos y) \, dy \right) \, dx \]

\[ = \int_1^2 \left[ xy + \sin y \right]_{y=3}^{y=4} \, dx \]

\[ = \int_1^2 (4x - 3x + \sin 4 - \sin 3) \, dx \]

\[ = \frac{x^2}{2} \bigg|_{x=1}^{x=2} + (\sin 4 - \sin 3) \cdot 1 \]

\[ = 2 - \frac{1}{2} + \sin 4 - \sin 3 \]
More general domains.

To define $\iint f(x,y) \, dA$

Take $R \supset D$

rectangle

Define $\overline{f}(x,y)$ on $R = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$

Then define

$$\iint f(x,y) \, dA := \iint \overline{f}(x,y) \, dA.$$ 

The interesting part: encoding the shape of $D$

into the limits of the iterated integral.
Call \( D \) \( x \)-simple if
(\text{or} \ y \text{-simple})

\[
\begin{align*}
\int_a^b f(x) \\ \int_a^b g(x)
\end{align*}
\]

\[
\int_a^b \int \, dy \\ dx
\]

\( a \leq x \leq b \)

\( f(x) \leq y \leq g(x) \)

\( f, g \) are cont. fu

\[
\begin{align*}
\int_a^b f(y) \\ \int_a^b g(y)
\end{align*}
\]

\( a \leq y \leq b \)

Most domains are both \( x \)- and \( y \)-simple

We will integrate only over \( x \)-simple or \( y \)-simple domains.

\( x \)-simple

\( y \)-simple

\( x \)-simple

\( y \)-simple

\( x \)-simple

\( y \)-simple

\( x \)-simple

\( y \)-simple
Example: Compute the volume of a tetrahedron in the first octant, bounded by the plane 
\[ z = 6 - 3x - 2y \]

Volume = volume under the graph of 
\[ z = f(x,y) = 6 - 3x - 2y \]
over the domain:

(he bottom face of our pyramid)

\[ V = \iiint_D (6 - 3x - 2y) \, dA \]

\[ = \int_0^2 \int_0^{\frac{6 - 2x}{3}} (6 - 3x - 2y) \, dy \, dx \]

\[ = \int_0^2 \left[ 6y - 3xy - 2y^2 \right]_{y=0}^{\frac{6 - 2x}{3}} \, dx \]

\[ = \int_0^2 \left( 12 - 4y - \frac{2}{3} (6 - 2y)^2 - \frac{2}{3} y (6 - 2y) \right) \, dx \]

\[ = \ldots \text{ finish.} \]

Also, the integral
\[ = \iint_D f(x,y) \, dy \, dx \]

Need the equation of this line.

The line 
\[ 6 - 3x - 2y = 0 \]
- intersection of the plane with the xy-plane.
When writing iterated integrals, the outside integral is only allowed to have numbers for limits. You can only use "x" in the limits inside the integral dx:

\[ \int_0^2 \int_0^{6 - \frac{3x}{2}} \, dy \, dx \]

\[ \text{x is used inside integral with respect to dx.} \]

If you write

\[ \int_0^x \int_{3}^{f(x,y)} \, dy \, dx \]

it doesn't make sense.

**Ex:**

\[ \int_0^1 \int_0^7 e^{y^2} \, dy \, dx \]

No formula for antiderivative!

\[ = \int_0^1 \int_0^7 e^{y^2} \, dx \, dy \]

\[ = \int_0^1 \left( e^{y^2} \cdot y \right) \, dy \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left( e^{y_i^2} \cdot y_i \right) \]

\[ = \text{use } u = y^2 \]

\[ = \frac{1}{2} \int_0^1 e^u \, du = \frac{1}{2} (e - 1) \]