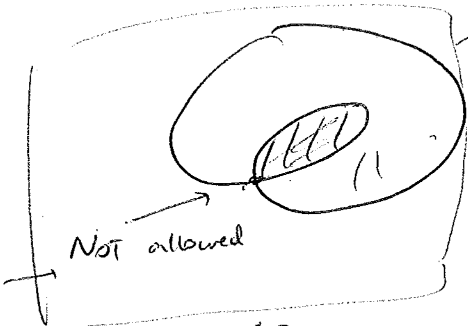
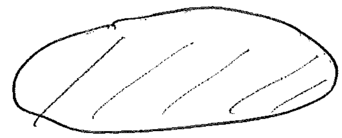
Last time: defined integral in $\mathbb{R}^2$ with resp. to area.
(Riemann sums).

Always assume: $f(x,y,z)$ is continuous
(piecewise-cont. at worst)

Domain is bounded by a simple curve
no self-intersections





book says: continuous, of finite length.

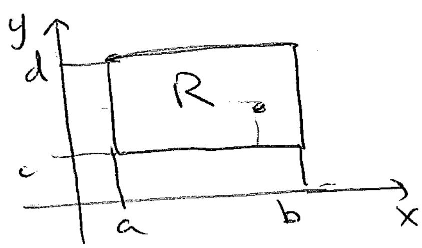
if we have a region like this, will break it into a union of simpler regions.

(we will simply assume smooth).

Key point of today:
To evaluate a double integral, you need to represent it as an iterated integral.

Ex: domain is a rectangle

$$R = [a,b] \times [c,d]$$



Claim: $\displaystyle\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$

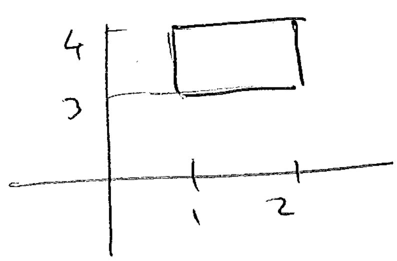
$\displaystyle = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx \right)}_{\text{for every fixed } y,\ \text{compute } \int_a^b f(x,y)\,dx \text{ in the usual way}} dy.$

(keep y as a parameter).

function of y.

a case of Fubini's Theorem.

Example: $R = [1,2] \times [3,4]$



Evaluate $\displaystyle\iint_R (x + \cos y)\,dA$

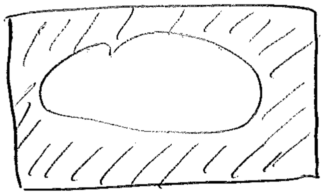
$\displaystyle = \int_1^2 \left(\int_3^4 (x + \cos y)\,dy \right) dx$

$\displaystyle = \int_1^2 (xy + \sin y)\Big|_{y=3}^{y=4} dx = \int_1^2 \underbrace{(4x - 3x)}_{x} + (\sin 4 - \sin 3)\,dx$

$\displaystyle = \frac{x^2}{2}\Big|_{x=1}^{x=2} + (\sin 4 - \sin 3)\cdot 1 = 2 - \frac{1}{2} + \sin 4 - \sin 3$

• More general domains.

To define $\iint_D f(x,y)\, dA$

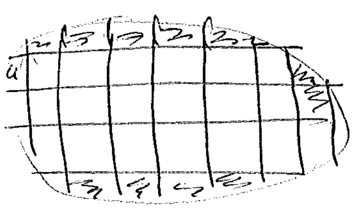


Take $R \supset D$ rectangle

Define $\tilde{f}(x,y)$ on R $= \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$

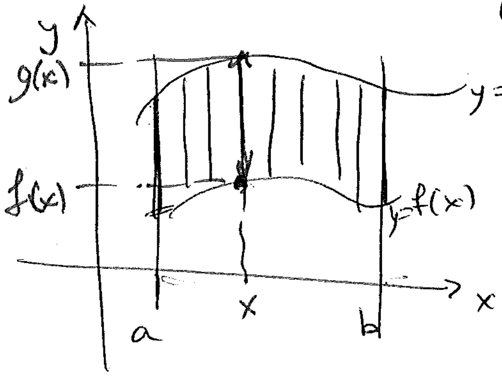
Then define

$$\iint_D f(x,y)\, dA := \iint_R \tilde{f}(x,y)\, dA.$$



The interesting part: encoding the shape of D into the limits of the iterated integral.

Call D x-simple (or y-simple) if



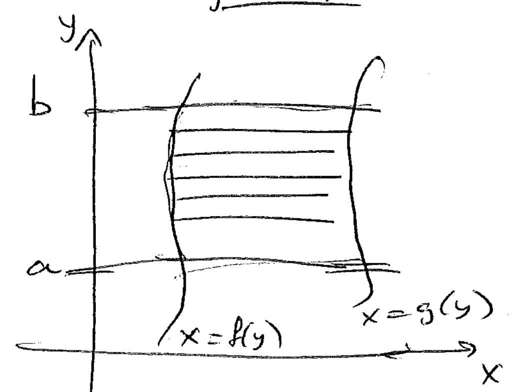
$$\int_a^b \int_{f(x)}^{g(x)} \sim \, dy \, dx$$

$a \le x \le b$

$f(x) \le y \le g(x)$

f, g are cont. fns

<u>y-simple</u>



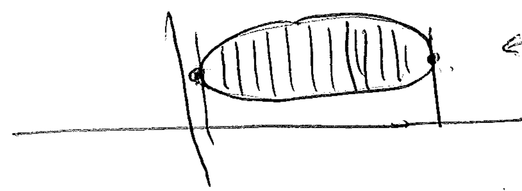
(maybe x-simple?)

$a \le y \le b$

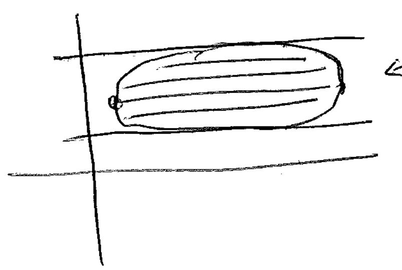
$f(y) \le x \le g(y)$

Most domains are both x- and y-simple

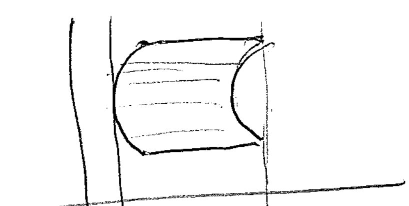
We will integrate only ~~to~~ over x-simple or y-simple domains.



← x-simple

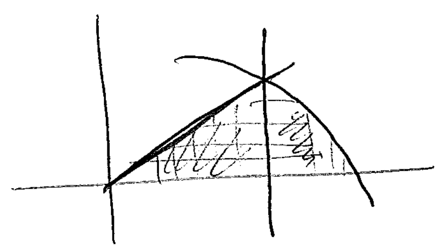


← y-simple.



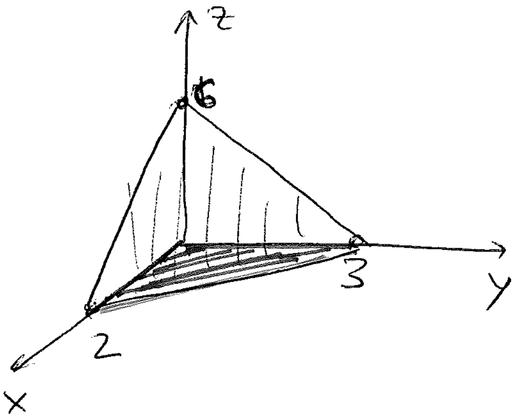
y-simple

<u>Ex</u>:



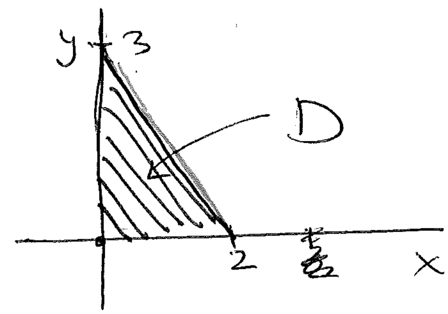
Example Compute the volume of the tetrahedron in the first octant, bounded by the plane

$$z = 6 - 3x - 2y$$

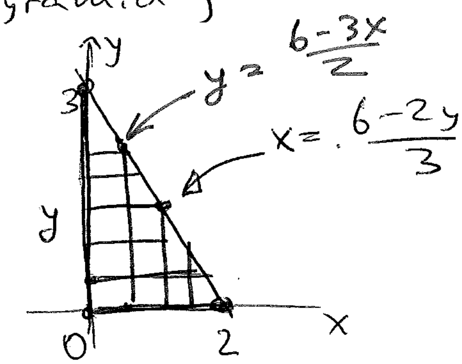


volume = volume under the graph of $z = f(x,y) = 6 - 3x - 2y$

over the domain:



(the bottom face of our pyramid)



Need the equation of this line.

The line $6 - 3x - 2y = 0$ - intersection of the plane with the xy-plane.

$$V = \iint_D (6 - 3x - 2y)\, dA$$

$$= \int_0^3 \int_0^{\frac{6-2y}{3}} (6 - 3x - 2y)\, dx\, dy$$

$$= \int_0^3 \left(6 \cdot \frac{6-2y}{3} - 3 \frac{x^2}{2} \Big|_{x=0}^{x=\frac{6-2y}{3}} - 2y \cdot \frac{6-2y}{3} \right) dy$$

$$= \int_0^3 \left(12 - 4y - \frac{3}{2}\left(\frac{6-2y}{3}\right)^2 - \frac{2}{3} y (6-2y) \right) dy$$

$= \ldots$ finish.

Also this integral

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} f(x,y)\, dy\, dx$$

When writing iterated integrals,

the outside integral is only allowed to have numbers for limits

you can use "x" in the limits only inside the integral dx:

$$\int_0^2 \int_0^{\frac{6-3x}{2}} dy\, dx$$

x is used

inside integral with respect to dx.

If you write

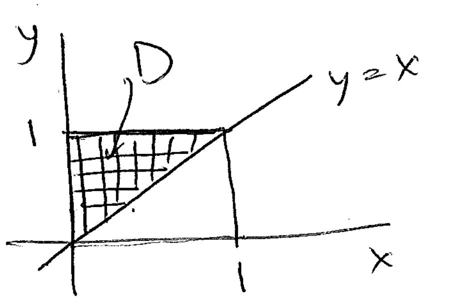
$$\int_0^x \int_1^3 f(x,y)\, dy\, dx$$

— it doesn't make sense.

EX: evaluate

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \iint_D e^{y^2}\, dA$$

No formula for antiderivative!



$$= \int_0^1 \int_0^y e^{y^2}\, dx\, dy$$

$$= \int_0^1 \left(e^{y^2}\cdot y\right) dy$$

$$= \text{use } u=y^2 \quad \frac{1}{2}\int_0^1 e^u\, du = \frac{1}{2}(e-1).$$