First print on Lagrange mult.

- For the method to work, we need to assume:
  - \( f(x, y, z) \) is differentiable
  - \( G(x, y, z) \) cont. diff.
  - \( \nabla f = \lambda \nabla G \), really works only when \( \lambda \neq 0 \).

So, when looking for candidate points for the extremum, make a list:

- critical points of \( f \) inside the domain
- on the boundary:
  1. use Lagrange mult.
     \[ F = f - \lambda G \]
     critical points of \( F \)
  2. All points where \( \nabla f \) does not exist
     (inside the domain or on the boundary)
  3. All points where \( \nabla G \) does not exist
  4. All points where \( \nabla G = 0 \).

If the boundary has "end points", these end points.

\[ \mathbb{R}^2, \quad D = \begin{cases} 3 \text{ different constraint functions} \\ 4 \text{ different Lagrange problems} \end{cases} \]
Example: Find \( \max f(x,y,z) \)

on a cube 

\[ 0 \leq x \leq 1 \]
\[ 0 \leq y \leq 1 \]
\[ 0 \leq z \leq 1 \]

Algorithm:

1. Find critical pts of \( f \) inside (or on the boundary) of the cube.
2. Find the candidate points on each face of the cube.
   (2 methods: Lagrange or plug in, e.g., \( x=1 \) if on the face \( x=1 \), work with a function of \( y,z \)).
3. Look for max on the edges
   (plug in \( x=1, y=0 \), for example --)
   (or for more complicated solids, use Lagrange with 2 constraints)

\[ f(x,y,z) = xG(k,y,z) - H(k,y,z) \]

\[ \text{equations of surface whose intersection is our edge.} \]

- Vertices
- points where \( f \) is not diff.
- points where \( JG, JH, \) are 0.
Linear programming: shortens this process in the situation where \( f(x_1, -x_n) \) is linear and the solid region is defined by linear inequalities \( \mathbf{a} \cdot \mathbf{x} = 0 \):

\[
\begin{align*}
 a_{11} x_1 + a_{12} x_2 + a_{13} x_3 & \geq 0 \\
 a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n & > 0 \\
 a_{m1} x_1 + \cdots + a_{mn} x_n & \geq 0
\end{align*}
\]

(convex polyhedron)

In fact, only need to check the vertices.

(Optimal reading.)

When more than one constraint:

The condition for an extremal point of \( f(x_1, -x_n) \) subject to constraints \( \mathbf{c}_1(x_1, -x_n) = 0 \)

\[ \mathbf{c}_m(x_1, -x_n) = 0 \]

is: \( \mathbf{Df} \) has to lie in the subspace of \( \mathbb{R}^n \) spanned by \( \mathbf{Dc}_1, \ldots, \mathbf{Dc}_m \) at that point,

\[
\mathbf{Df} = \lambda_1 \mathbf{Dc}_1 + \cdots + \lambda_m \mathbf{Dc}_m
\]

Lagrange multipliers
I want to minimize (maximize) \( f(x, y, z) \) when \( (x, y, z) \in C \).

\[
\nabla f = \lambda \nabla C_1 + \mu \nabla C_2
\]

**Meaning of \( \lambda \):** it tells you how much \( f \) will change if you replace \( C(x, y, z) = 0 \) with \( C(x, y, z) = h \) for small \( h \).

\[
\max f \text{ will change by } \lambda \cdot |\nabla C_1|
\]

(see p. 773–774)
Integrals over rectangles $\mathbb{R}^2$

want to define:
\[
\iint_D f(x,y) \, dA
\]
double integral
(number of integrals = dimension of the domain)
(it is a single symbol).

Note: there will be no "indefinite integral".

\[(1) \quad \iint_D f(x,y) \, dA\]
\[
\int_D \int f(x,y) \, dA
\]

Don't make sense.

The correct usage:
\[
\int_D f(x,y) \, dA
\]

1. Let $D$ be a rectangle.

Riemann sum:

\[ a = f(x,y) \]

subdivide our rectangle $D$

pick a point in each:

$$(x^*, y^*)$$
Riemann sum for the double integral is

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij}) \Delta x_i \Delta y_j = \int_{P} \int_{D} f(x, y) \, dA \]

where \( \Delta x_i \) and \( \Delta y_j \) are the areas of the small rectangles.

Approximates the volume under the graph of \( f(x, y) \) over our rectangle \( D \).

*Theorem:* \( \Delta x_i^2 + \Delta y_j^2 \rightarrow 0 \) then all subdivisions will give Riemann sums that are very close to the volume under the graph.

Let \( \varepsilon > 0 \) s.t. if for every \( i, j \),

\( \text{diam} \left( R_{ij} \right) < \delta \)

\[ |R_P - R_{P'}| < \varepsilon \]

for any \( P, P' \) satisfying this and any choice \((x_{ij}, y_{ij})\).

Then their limit exists, and is called \( \int_{D} \int f(x, y) \, dA \).

(Assume \( f \) is cont. on \( D \)).