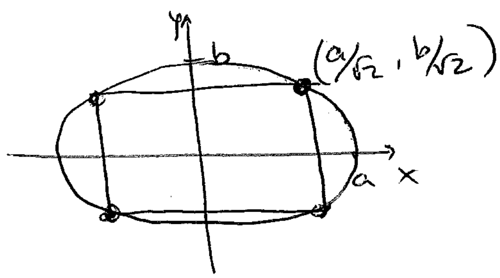
Lagrange multipliers

- finding max/min of $f(x,y)$ when (x,y) are subject to constraint $G(x,y)=0$.

(General: $f(x_1, \ldots x_m)$ when
$$G_1(x_1 \ldots x_m) = 0$$
$$G_2(x_1 \ldots x_m) = 0$$
$$\vdots$$
$$G_n(x_1 \ldots x_m) = 0$$
)

Example: max area of a rectangle (parallel to the axes)



that fits into the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

want: $\max(xy)$ as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\underset{f(x,y)}{\underset{\parallel}{xy}} \qquad G(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

Trick: make a new function

$$F(x,y,\lambda) := f(x,y) - \lambda G(x,y)$$
$$= xy - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$\uparrow$ (λ) Lagrange multiplier

Then look for the ~~[illegible]~~ critical points of $F(x,y,\lambda)$: Find solutions to the system of equations $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial \lambda} = 0$.

In our example, get:

$$\begin{cases} y - 2x\lambda \cdot \frac{1}{a^2} = 0 & \leftarrow \partial F/\partial x \\ x - 2y\lambda \cdot \frac{1}{b^2} = 0 & \leftarrow \partial F/\partial y \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 & \leftarrow \partial F/\partial \lambda \end{cases}$$

— should be the constraint.

Solve for (x, y, λ)

don't care.

$$\begin{cases} x = \frac{2y\lambda}{b^2} \\ y = \frac{2x\lambda}{a^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}$$

$\lambda \neq 0$
(otherwise $x = y = 0$, doesn't fit the last equ.)

$$\begin{cases} \frac{x}{y} = \frac{y}{x} \frac{a^2}{b^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}$$

$$\begin{cases} x^2 = y^2 \cdot \frac{a^2}{b^2} \\ \frac{y^2 a^2/b^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}$$

$2\frac{y^2}{b^2} = 1$

$x = \pm a/\sqrt{2}$

$y = \pm b/\sqrt{2}$

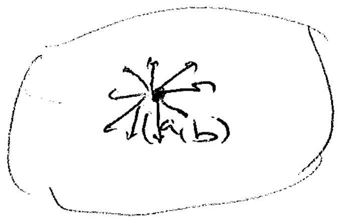
<u>Answer</u>: max area $= 4 \frac{ab}{2}$, occurs at $\left(\pm\frac{a}{\sqrt{2}}, \pm\frac{b}{\sqrt{2}}\right)$

$= 2ab$

one of the vertices.

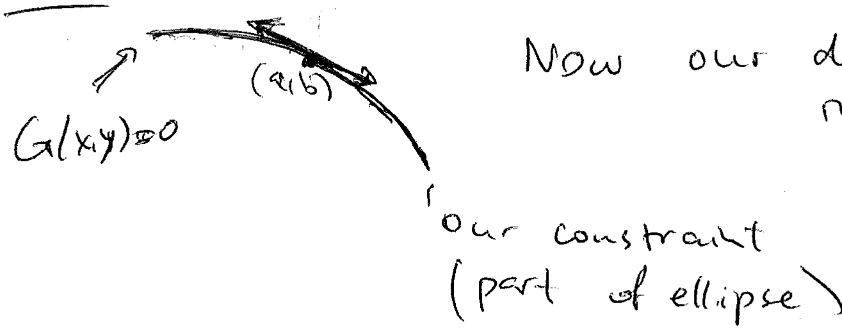
Why it works: f is differentiable.

Recall: max/min can occur (inside the domain) only at a critical point.



all directional derivatives at (a,b) should be 0.
(if one is positive, we could go in that direction and $f(x,y)$ would get bigger).

Now:



Now our directional derivatives in the directions tangent to the constraint should be 0.
(otherwise could walk a bit in one of the directions along the curve and $f(x,y)$ would get bigger).

This says, $\bar{\nabla}f_{(a,b)} \perp \bar{u}$

$\bar{u}$ tangent to the curve $G(x,y)=0$ at (a,b)
(i.e. perpendicular to $\bar{\nabla}G$ at (a,b)).

Conclusion:
if (a,b) is a local max or min for $f(x,y)$ as $G(x,y)=0$
then $\bar{\nabla}f_{(a,b)}$ is parallel to $\bar{\nabla}G_{(a,b)}$.

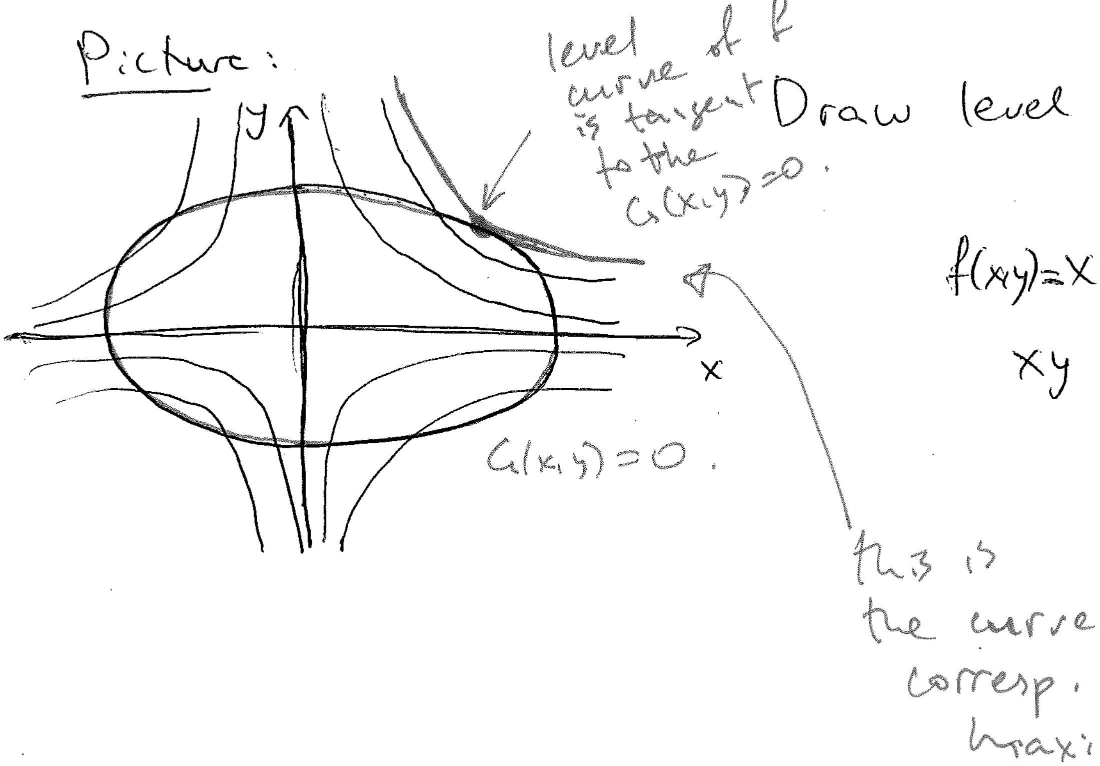
So there exists λ s.t. $\bar{\nabla} f = \lambda \bar{\nabla} G$ at (a,b).

We were considering

$$F(x,y,\lambda) = f(x,y) - \lambda G(x,y).$$

$$\bar{\nabla} F = 0 \quad \text{if and only if} \quad \bar{\nabla} f = \lambda \bar{\nabla} G.$$

Picture:



Draw level curves of $f(x,y)$.

$f(x,y) = xy$. level curves: $xy = c$ – hyperbolas

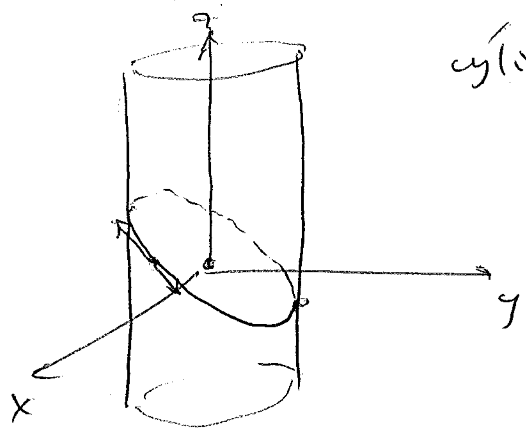
In $\mathbb{R}^3$: same picture, but instead of level curves have level surfaces.

max/min $f(x,y,z)$ subject to $G(x,y,z) = 0$

Again: look for the list of points where $\bar{\nabla} f \parallel \bar{\nabla} G$, which means, find critical points of: $f(x,y,z) - \lambda G(x,y,z) =: F(x,y,z,\lambda)$.

Could have two constraints:

Example: $x^2+y^2=1$, intersect it with the
cylinder
plane $x+3y-z=0$



– get an ellipse.

Find the points on this ellipse closest and farthest away from the origin.

We had: one constraint
$\bar{\nabla} f \parallel \bar{\nabla} G$

my function

$f(x,y,z) = x^2+y^2+z^2$
= square of distance from (0,0)

Now, two constraints, $G(x,y,z)=0$
$H(x,y,z)=0$.

Want: $\bar{\nabla} f$ has to be perpendicular to the tangent line to this ellipse.

$\bar{\nabla} f$ has to lie in the plane spanned by $\bar{\nabla} G$ and $\bar{\nabla} H$.

Expressed as: $\bar{\nabla} f = \lambda \bar{\nabla} G + \mu \bar{\nabla} H$

two Lagrange multipliers.

Recipe:

make $F(x, y, z, \lambda, \mu) =$

$$f(x,y,z) - \lambda G(x,y,z) - \mu H(x,y,z)$$

look for critical points.

Example: $(x^2+y^2+z^2) - \lambda(x^2+y^2-1) - \mu(x+3y-z)$

$$\begin{cases} 2x - 2x\lambda - \mu = 0 \\ 2y - 2y\lambda - 3\mu = 0 \\ 2z + \mu = 0 \\ x^2+y^2-1=0 \\ x+3y-z = 0 \end{cases}$$

Homework: solve for (x, y, z).