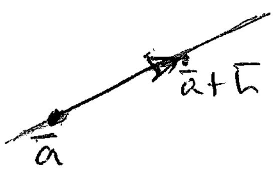
Today: critical points, Lagrange
(13.1) (13.3)

Last time:

Taylor series in 2 variables:

Hessian

$$f(\bar{a}+\bar{h}) = f(\bar{a}) + \nabla f|_a \cdot \bar{h} + \underbrace{\tfrac{1}{2}\bar{h}^T H|_a \cdot \bar{h}}_{[h_1, h_2]\begin{bmatrix} f_{xx}(a) & f_{xy}(a) \\ f_{yx}(a) & f_{yy}(a)\end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}$$



(Reminder of 12.9)

$$\underbrace{+\tfrac{1}{6}(\nabla\cdot\bar{h})^3(a) + \cdots}_{\text{higher-order terms}}$$

What does it have to do with critical points?

- assume $f(x,y)$ has continuous 2nd order partial derivatives at (a,b).

  Suppose (a,b) is a critical point.
  (i.e. $\nabla f|_{(a,b)} = 0$) Also get: $D_u f|_{(a,b)} = 0$ for every $\bar{u}$,
  so the tangent plane at (a,b) is horizontal.

- What can the graph of f look like?
  Use the secont term of the approximation.

At $(a+h_1, b+h_2)$, we have

$$f(a+h_1, b+h_2) \approx f(a,b) + \frac{1}{2}\underbrace{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}$$

$$h_1(f_{xx} \cdot h_1 + f_{xy} \cdot h_2) + h_2(f_{yx} \cdot h_1 + f_{yy} \cdot h_2)$$

$$= h_1^2 \cdot f_{xx}(a,b) + 2h_1h_2 f_{xy}(a,b) + h_2^2 f_{yy}(a,b).$$

So our graph is approximated by

$$z = (h_1^2 \cdot c_1 + h_1h_2 \cdot c_2 + h_2^2 \cdot c_3) + f(a,b)$$

constants.

$c_1 = f_{xx}(a,b)$, $c_3 = f_{yy}(a,b)$

$c_2 = 2f_{xy}(a,b)$

Can we classify graphs of quadratic polynomials? — 10.6.

classify any quadratic surfaces given by

$$c_1 z^2 + c_2 x^2 + c_3 y^2 + c_4 xy + \text{—} + c_6 xz + d = 0$$

reduce to a few cases. (allowed linear changes of coordinates).

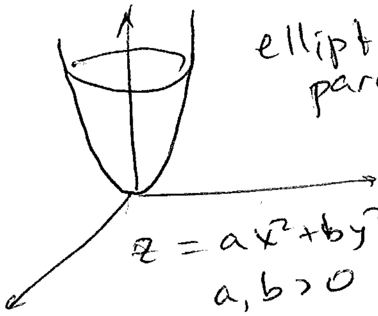
Relevant for us:

$z = ax^2 + by^2$, $a, b > 0$ ~~[illegible]~~

$z = ax^2 - by^2$ $a, b > 0$.

elliptic paraboloid

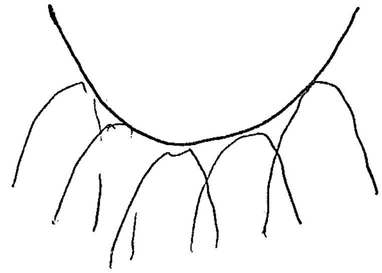
hyperbolic paraboloid.



elliptic paraboloid

$z = ax^2 + by^2$

$a, b > 0$

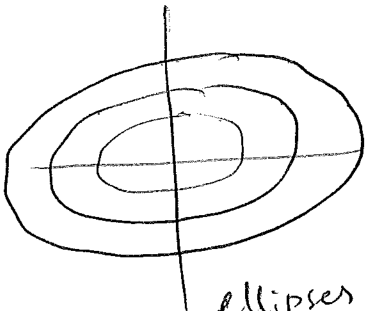


hyperbolic

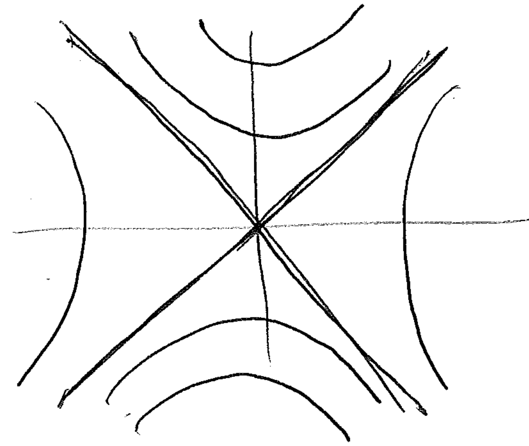
$z = ax^2 - by^2$

$a, b > 0.$

level curves:



ellipses



hyperbolas

$\det \underset{p}{H} > 0$

Hessian

$H = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$

$\det H < 0.$

$\begin{pmatrix} 2a & 0 \\ 0 & -2b \end{pmatrix}$

$\frac{\partial z}{\partial x} = 2ax$

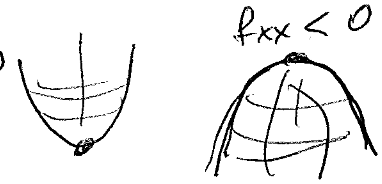
$\frac{\partial z}{\partial y} = 2by$

$\frac{\partial^2 z}{\partial x^2} = 2a$

This is why det (Hessian) can be used to classify the critical points!

det $H > 0$, then around (a, b) the graph looks like elliptic paraboloid

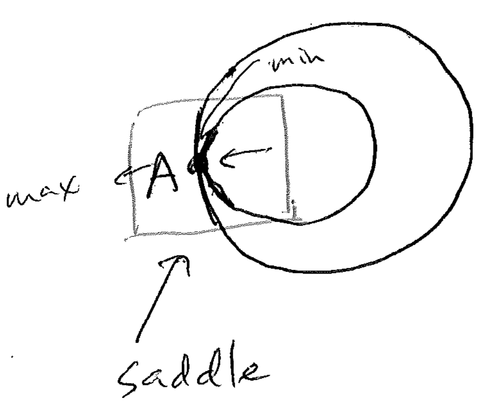
$f_{xx} > 0$ $f_{xx} < 0.$



— max / min.

When det $H < 0$, it is like hyperbolic paraboloid — saddle point

Webwork had :



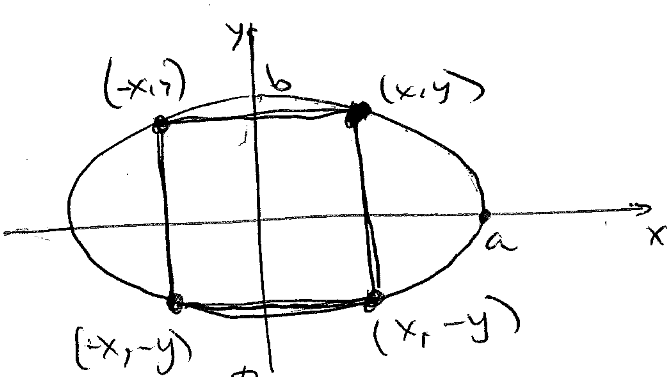
Lagrange multipliers

Find the max/min (extreme) values of $f(x,y)$ (or $f(x,y,z)$ ...) or $f(x_1, \ldots x_n)$

on a closed bounded domain.

<u>Example</u> find the area of the max rectangle with sides parallel to the axes that fits in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

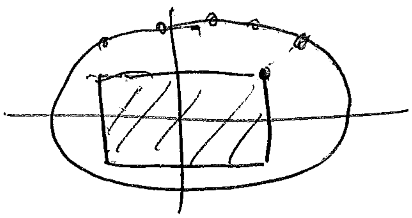
(a, b = constants, given).



$A = 4xy$

Then one vertex (x,y) determines the rest.

It's clear we should let the vertices be on the boundary.



We want to maximize

$$f(x,y) = xy$$

on the domain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1.$$

(vertex is inside or on the ellipse).

Really, only need to consider $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

One way: parametrize the ellipse:

$$x = a\cos t$$
$$y = b\sin t \qquad 0 \le t \le 2\pi$$

Plug this into our $f(x,y)$.

Get: $h(t) = f(x,y) = ab\cos t \sin t$.

Now use single-variable to find max $h(t)$ as $0 \le t < 2\pi$.

Better way: Lagrange multipliers: 13.3.

And Read 13.2