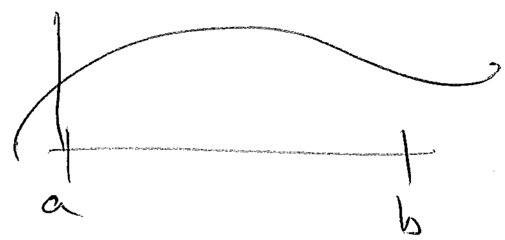
Today: critical points (13.1 and 12.9)
Taylor approximations

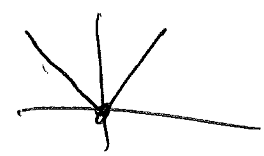
There will be Webwork due Thursday evening.

Extreme (max or min) value of a continuous function:

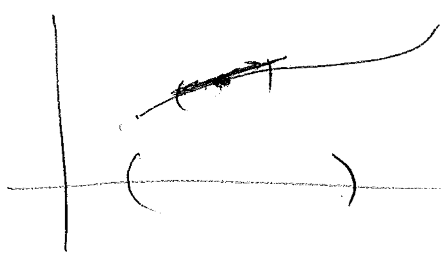


in 1 variable,
can occur at:

- critical points: $f'(x) = 0$
- points where f is not differentiable:

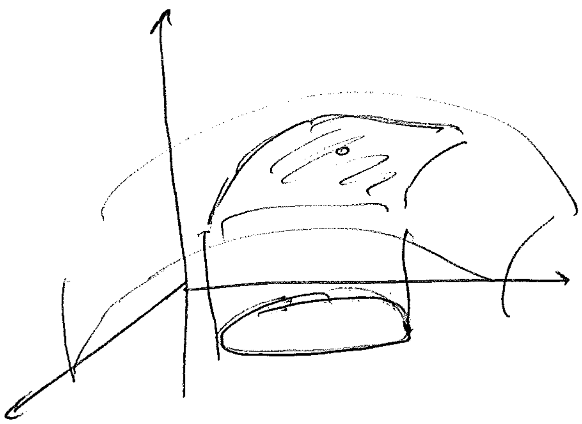


- end points of the interval.



In several variables:

FACT: - a continuous function on a closed bounded domain also always attains its max and min values.



Where can this max/min occur?

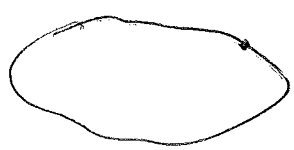
(1) at a critical point

before: $f'(a)=0$

Now: $\begin{cases} \left.\dfrac{\partial f}{\partial x}\right|_{(a,b)} = 0 \\ \left.\dfrac{\partial f}{\partial y}\right|_{(a,b)} = 0 \end{cases}$

(2) or at a point where f is not differentiable

(3) at ~~the~~ a point on the boundary of the domain.



← coming next class.

Today: critical points:

Example: $f(x,y) = x^3 + x^2y^2 - y^4$.

Find its critical points.

Want to solve: $\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$

compute $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2$$

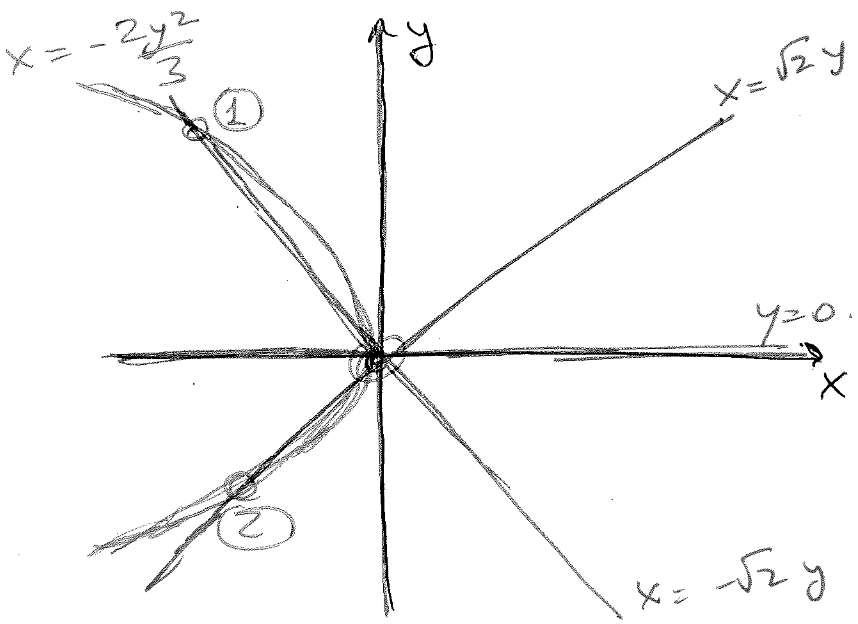
$$\frac{\partial f}{\partial y} = 2x^2y - 4y^3$$

Get: $\begin{cases} 3x^2 + 2xy^2 = 0 \\ 2x^2y - 4y^3 = 0 \end{cases}$

No general method! (use computers).

Solving: $\begin{cases} 3x^2+2xy^2 = x(3x+2y^2) = 0 \quad \text{(red)} \\ 2x^2y - 4y^3 = y(2x^2-4y^2) \\ \qquad = 2y(x^2-2y^2) = 0. \quad \text{(blue)} \end{cases}$

Let us draw the set of solutions to each equation:



$3x + 2y^2 = 0$

$x = \dfrac{-2y^2}{3}$

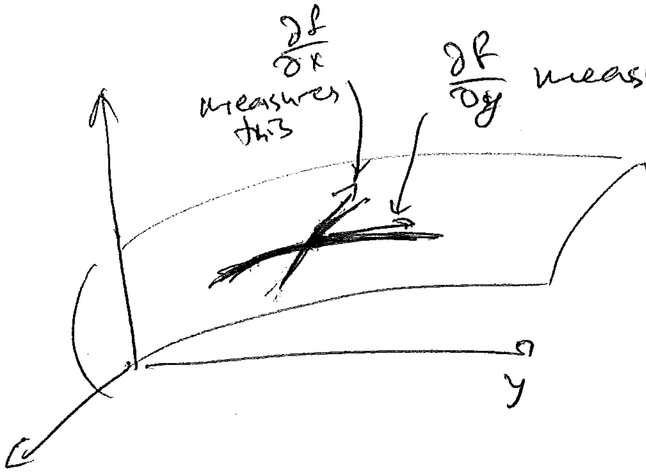
$x^2 - 2y^2 = 0$

$x = \pm\sqrt{2}y$

Critical points: intersection of the red and blue sets.

Aside.

About geometry:



$\frac{\partial f}{\partial y}$ measures slope of this.

So the points where $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$ are the points where someone walking East-West (along x) on the graph has a chance of hitting local max/min.

Example, continued:

In our example, the critical points are:

$(0,0)$ and: ① $\begin{cases} x = -\sqrt{2}y \\ x = -\frac{2y^2}{3} \end{cases}$ $\Leftrightarrow$ $-\cancel{\sqrt{2}}y = -\cancel{2}^{\sqrt{2}}\frac{y^2}{3}$

$y = 0$ or $y = 3/\sqrt{2}$

②: $\begin{cases} x = \sqrt{2}y \\ x = -\frac{2y^2}{3} \end{cases}$ $y = -3/\sqrt{2}$ or $y = 0$

Get: $\boxed{(0,0),\ \left(-3, \frac{3}{\sqrt{2}}\right),\ \left(-3, -\frac{3}{\sqrt{2}}\right)}$

Second derivative test:

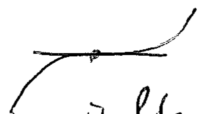
- about finding local max/min.

Suppose (a,b) is a critical point.
How can we tell if it is a local $\left\{ \begin{array}{l} \text{max} \\ \text{min} \\ \text{neither?} \end{array} \right.$

(Recall single variable:


max


min

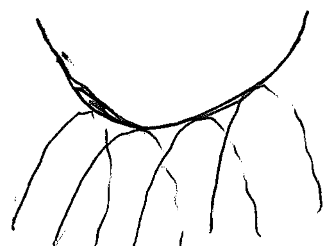

inflection

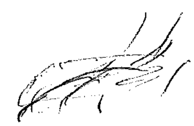
- used second derivative test.

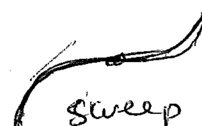
In 2 variables:


local max


min


saddle point


neither


sweep along the other axis

'second derivative test'

in 2 variables:

$$H = \underbrace{\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}}_{\text{the Hessian}}$$

— if the second partials are continuous, then this a symmetric ~~matrix~~

$$\det H = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

In our example: take $(-3, 3/\sqrt{2})$:

we had: $\frac{\partial f}{\partial x} = 3x^2 + 2xy^2$

$\frac{\partial f}{\partial y} = 2x^2y - 4y^3$

$\frac{\partial^2 f}{\partial x^2} = 6x + 2y^2$

$\frac{\partial^2 f}{\partial x \partial y} = 4xy$

$\frac{\partial^2 f}{\partial y^2} = 2x^2 - 12y^2$

Plug in $(-3, 3/\sqrt{2})$.

$$H = \begin{bmatrix} -18+9 & -12 \cdot 3/\sqrt{2} \\ -\frac{12 \cdot 3}{\sqrt{2}} & 18 - 12 \cdot \frac{9}{2} \end{bmatrix}$$

$\det H < 0$

If $\det H > 0 \Rightarrow$ local min $\leftarrow f_{xx} > 0$

max $\leftarrow f_{xx} < 0$

$\det H < 0 \Rightarrow$ saddle point

$\det H = 0$: no information. $\leftarrow$ (in our example: $(0,0)$ is this kind)

Why: please Read 13.1

Hint — uses 12.9.