Today: critical points
Taylor approximations

There will be Webwork due Thursday evening.

Extreme (max or min) value of a continuous function:

in 1 variable, can occur at:

- critical points: \( f'(x) = 0 \)
- points where \( f \) is not differentiable
- end points of the interval
In several variables:
- a continuous function on a closed bounded domain also always attains its max and min values.

Where can this max/min occur?

1. at a critical point before: \( f'(a) = 0 \)
   Now: \( \frac{df}{dx} |_{(a,b)} = 0 \)
   \( \frac{df}{dy} |_{(a,b)} = 0 \)

2. or at a point where \( f \) is not differentiable

3. at a point on the boundary of the domain.

← coming next class.
Today: critical points:

Example: \( f(x,y) = x^3 + x^2y^2 - y^4 \).

Find its critical points.

Want to solve:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 0 \\
\frac{\partial f}{\partial y} &= 0
\end{align*}
\]

Compute \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \):

\[
\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 \\
\frac{\partial f}{\partial y} = 2x^2y - 4y^3
\]

Get:

\[
\begin{align*}
3x^2 + 2xy^2 &= 0 \\
2x^2y - 4y^3 &= 0
\end{align*}
\]

No general method! (Use computers).

Solving:

\[
\begin{align*}
3x^2 + 2xy^2 &= x(3x + 2y^2) = 0. \quad \text{(red)} \\
2x^2y - 4y^3 &= y(2x^2 - 4y^2) = 0. \quad \text{(blue)}
\end{align*}
\]

Let us draw the set of solutions to each equation:

Critical points: intersection of the red and blue sets.
About geometry:

\[ \frac{df}{dx} \text{ measures slope at } x. \]
\[ \frac{df}{dy} \text{ measures slope at } y. \]

So the points where \( \frac{df}{dx}(a,b) = 0 \) are the points where someone walking East-West (along x) on the graph has a chance of hitting local max/min.

Example, continued:

In our example, the critical points are:

\( (0,0) \) and:

1. \[
\begin{align*}
    x &= -\sqrt{2}y \\
    x &= -2\frac{y^2}{3} \\
\end{align*}
\]
   \( \Rightarrow -\sqrt{2}y = -2\frac{y^2}{3} \)
   \( \Rightarrow y = 0 \)
   \( \text{or } y = 3\sqrt{2} \)

2. \[
\begin{align*}
    x &= \sqrt{2}y \\
    x &= -2\frac{y^2}{3} \\
\end{align*}
\]
   \( \Rightarrow \frac{\sqrt{2}y}{3} = -2\frac{y^2}{3} \)
   \( \Rightarrow y = -\frac{3}{\sqrt{2}} \)
   \( \text{or } y = 0 \)

Get: \( (0,0), \left(-3, \frac{3}{\sqrt{2}}\right), \left(-3, -\frac{3}{\sqrt{2}}\right) \)
Second derivative test:
- about finding local max/min.
  Suppose $(a,b)$ is a critical point.
  How can we tell if it is a local max/min, neither?

(Recall single variable):

- Max
- Min
- Inflection

In 2 variables:

- Local max
- Min
- Saddle point
- Sweep along the other axis

Used second derivative test.
Second derivative test \\

In 2 variables:

\[ H = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix} \]

The Hessian

\[ \det H = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \]

In our example: take \((-3, \frac{3}{\sqrt{2}})\):

we have:

\[ \frac{\partial f}{\partial x} = 3x^2 + 2xy^2 \]
\[ \frac{\partial f}{\partial y} = 2x^2y - 4y^3 \]
\[ \frac{\partial^2 f}{\partial x^2} = 6x + 2xy^2 \]
\[ \frac{\partial^2 f}{\partial x \partial y} = 4xy \]
\[ \frac{\partial^2 f}{\partial y^2} = 2x^2 - 12y^2 \]

Plug in \((-3, \frac{3}{\sqrt{2}})\)

\[ H = \begin{pmatrix}
-18 + 9 & -12 \cdot \frac{3}{\sqrt{2}} \\
-12 \cdot \frac{3}{\sqrt{2}} & 18 - 12 \cdot \frac{9}{2}
\end{pmatrix} \]

\[ \det H < 0 \]

If \( \det H > 0 \) => local min. \( f_{xx} > 0 \)

max \( f_{xx} < 0 \)

\( \det H < 0 \) => saddle point

\( \det H = 0 \) : no information

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Why? please Read 13.1

Hint - use 12.9.