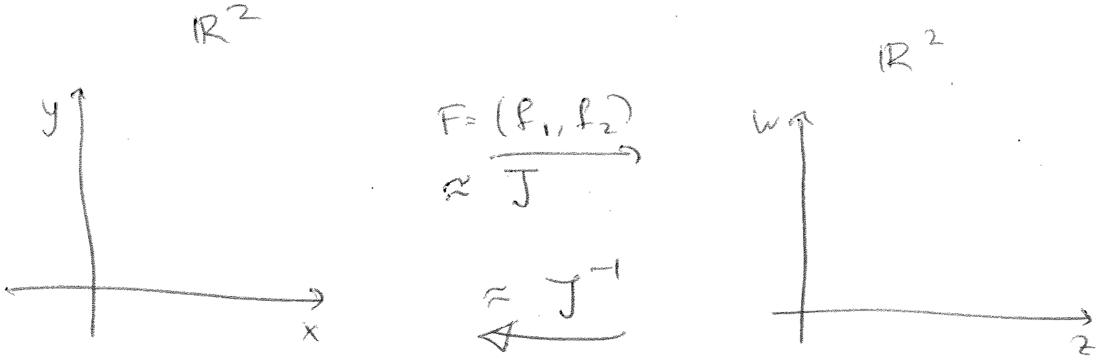
Today: <u>Implicit Function Thm</u> (the end)

<u>Last</u>: stopped at the Inverse Function Thm:

$$\begin{cases} z = f_1(x,y) \\ w = f_2(x,y) \end{cases}$$ we discussed how to solve for $\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \ldots$



When does this exist?

When can we express x, y as differentiable functions of z, w?

And if we can, find $\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}$.

we used linearization of our transformation:

$$\begin{bmatrix} z \\ w \end{bmatrix} \approx \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}}\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

evaluated at (x_0, y_0)

if this was $=$

then this would be a linear equation.

Then could solve for x, y in terms of z, w.

provided the Jacobian: $\det\begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} \neq 0$.

Inverse function theorem says:

if Jacobian $\neq 0$, then (x, y) can be expressed as a function of (z, w) in a neighbourhood of (z_0, w_0); and this function is differentiable, and its partial derivatives are given by the inverse of the Jacobian matrix.

Implicit Function Thm

More general:

$$\begin{cases} F(x,y,z,w) = 0 \\ G(x,y,z,w) = 0 \end{cases}$$

(**Note:** inverse function was a special case:

$F = f_1(x,y) - z$

$G = f_2(x,y) - w$

f_1, f_2 from the previous page.)

Example:

$$\begin{cases} xy + xz + zw + w^2 - 4 = 0 \\ w^2 y + x = 0 \end{cases}$$

Can choose 2 dependent, 2 indep. variables.
(Number of dependent variables should equal the number of equations).

In our example: let's say z, w will be dependent; x, y - independent.

Try to solve for $\frac{\partial z}{\partial x}, \frac{\partial w}{\partial x}$

(could also do that for $\frac{\partial z}{\partial y}, \frac{\partial w}{\partial y}$)

(This system of equations defines z, w as implicit functions of x, y).

/ Recall: $F(x,y,z) = 0$ ← single equation defines z as an implicit function of x, y. /

I will find $\frac{\partial z}{\partial x}, \frac{\partial w}{\partial x}$ in the example.

Differentiate both equations with respect to x.

$$\begin{cases} y + \underbrace{\left(z + x\frac{\partial z}{\partial x}\right)}_{\text{from } zx} + \underbrace{\left(z\frac{\partial w}{\partial x} + \frac{\partial z}{\partial x}\cdot w\right)}_{\text{from } zw} + 2w\frac{\partial w}{\partial x} = 0 \\ y \cdot 2w\frac{\partial w}{\partial x} + 1 = 0 \end{cases}$$

Two unknowns: $\frac{\partial z}{\partial x}, \frac{\partial w}{\partial x}$. Two linear equations.

Solve:

$$\begin{cases} \frac{\partial z}{\partial x}(x+w) + \frac{\partial w}{\partial x}(z+2w) = -(y+z) \\ 0 \cdot \frac{\partial z}{\partial x} + 2yw\frac{\partial w}{\partial x} = -1 \end{cases}$$

Can use Cramer's rule now

Matrix form:

$$\begin{bmatrix} x+w & z+2w \\ 0 & 2yw \end{bmatrix} \begin{bmatrix} \partial z/\partial x \\ \partial w/\partial x \end{bmatrix} = \begin{bmatrix} -(y+z) \\ -1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} \partial z/\partial x \\ \partial w/\partial x \end{bmatrix} = \begin{bmatrix} x+w & z+2w \\ 0 & 2yw \end{bmatrix}^{-1} \begin{bmatrix} -(y+z) \\ -1 \end{bmatrix}$$

General form:

$$\begin{cases} F(x,y,z,w) = 0 \\ G(x,y,z,w) = 0 \end{cases}$$

$\frac{\partial}{\partial x}$:
$$\begin{cases} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial F}{\partial w} \cdot \dfrac{\partial w}{\partial x} = 0 \\[2ex] \dfrac{\partial G}{\partial x} + \dfrac{\partial G}{\partial z} \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial G}{\partial w} \cdot \dfrac{\partial w}{\partial x} = 0 \end{cases}$$

Get:
$$\begin{bmatrix} \partial z/\partial x \\ \partial w/\partial x \end{bmatrix} = \underbrace{\begin{bmatrix} \partial F/\partial z & \partial F/\partial w \\ \partial G/\partial z & \partial G/\partial w \end{bmatrix}}_{\text{Jacobian: (with respect to dependent variables)}}^{-1} \cdot \begin{bmatrix} -\partial F/\partial x \\ -\partial G/\partial x \end{bmatrix}$$

(it replaces the condition
$(1\times 1) \to F_z \neq 0$ when $F(x,y,z) = 0$).

(always a square matrix of size = number of eqns = number of dependent variables)

Has to have $\neq 0$ determinant.

Implicit Fn: det Jac $\neq 0$ at (x_0, y_0) guarantees that (z, w) can be thought of as implicit functions of (x, y) and are differentiable near ~~...~~ (x_0, y_0)
