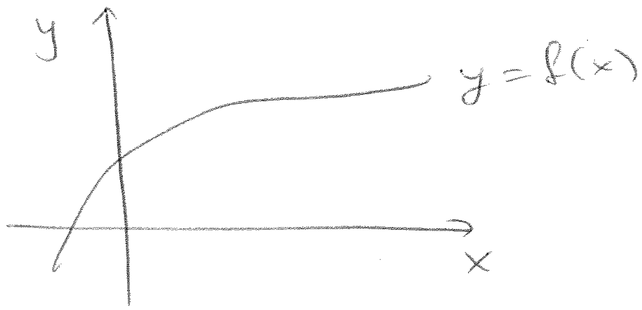
Today: inverse function Theorem
and Implicit function Theorem

Inverse Function Theorem:

- one variable



Want to solve for x in terms of y.

Get $x = f^{-1}(y)$

the inverse function.

(rename the variable, $f^{-1}(y)$, might as well write $f^{-1}(x)$)

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(x)}$$

careful: which point to evaluate at.

Compare this with implicit differentiation:

$$y = f(x)$$

What we want to find is the derivative of the inverse function, which ~~means~~ means, we want $\frac{dx}{dy}$.

$$\frac{dy}{dx} = f'(x)$$

$$x = f^{-1}(y)$$

then $\frac{dx}{dy} = \left(f^{-1}\right)'(y)$.

Take the equation $y = f(x)$

differentiate with respect to y implicitly.

$$\frac{d}{dy}\Big(y = f(x) \Big)$$

Get:

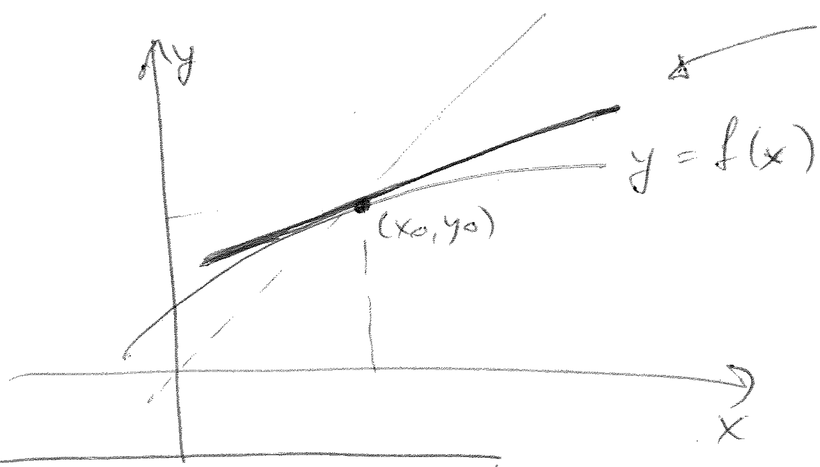
$$1 = \underset{\underset{\frac{df}{dx}}{\shortparallel}}{f'(x)} \cdot \frac{dx}{dy}$$

$$\boxed{\frac{dx}{dy} = 1 \Big/ \frac{df}{dx}}$$

ex: $f(x) = e^x$

$y = e^x$ $\qquad\qquad x = \ln(y)$

$\dfrac{df}{dx} = \underset{\underset{y}{\shortparallel}}{e^x}$ $\qquad\qquad \dfrac{d}{dy}(\ln(y)) = \dfrac{1}{y}$



$\left.\dfrac{df}{dx}\right|_{x_0}$ measures the slope of this line

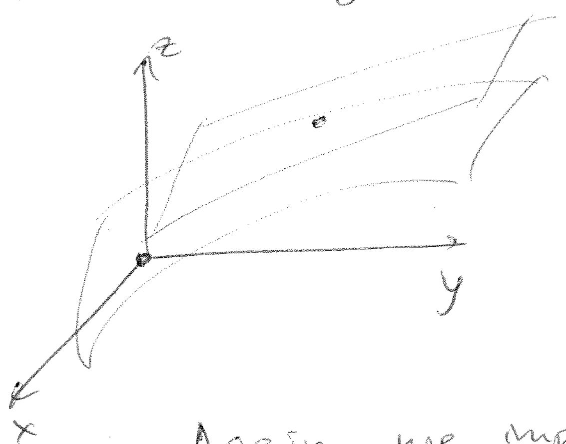
To get inverse function, flip about the line $x=y$.

$\dfrac{d(f^{-1}(y))}{dy}$ measures the slope of the same line but the axes are switched.

All of this works as long as $f'(x_0) \neq 0$.

Now in several variables

Let $z = f(x,y)$



Suppose want to solve for x in terms of y, z.

We can ask for the partial derivatives:

$$\frac{\partial x}{\partial z}, \quad \frac{\partial x}{\partial y}.$$

Again, use implicit differentiation to find them.

$$\underline{z = f(x,y)}$$

$$\frac{\partial}{\partial z}: \quad 1 = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial z}}_{=0}$$

$$\frac{\partial}{\partial y}: \quad 0 = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial f}{\partial y} \cdot 1$$

$$\boxed{\frac{\partial x}{\partial z} = 1 \Big/ \frac{\partial f}{\partial x} \qquad \frac{\partial x}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial x}}$$

Approximate our $f(x,y)$ by its linear approximation. Imagine that $f(x,y)$ was linear.

$z = ax + by$. solve for x in terms of y, z.

$$x = \frac{z - by}{a} = \frac{z}{a} - \frac{by}{a}.$$

$$\frac{\partial x}{\partial z} = \frac{1}{a}, \quad \frac{\partial x}{\partial y} = \frac{-b}{a}.$$

Inverse function Theorem says:

$z = f(x,y)$ (for functions from $\mathbb{R}^2 \to \mathbb{R}$)

you can solve for x as a function of (y,z)

if $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} \neq 0$

If f is continuously differentiable and $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} \neq 0$

Then in a neighbourhood of (x_0, y_0),
the solution exists and is a differentiable
function of y, z.

More general version:

ex: $f : \underset{(x,y)}{\mathbb{R}^2} \to \underset{(z,w)}{\mathbb{R}^2}$

$$\begin{cases} z = f_1(x,y) \\ w = f_2(x,y) \end{cases}$$

Then can expect to solve for (x,y) in terms of (z,w).

When does the inverse function exist?

- consider the linear approximation:

Recall: Jacobian matrix: $\begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix}$

$$\begin{bmatrix} z \\ w \end{bmatrix} \approx \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$z_0 = f_1(x_0, y_0)$

$w_0 = f_2(x_0, y_0)$

(the first line = linear approx. for f_1

the second line = linear approx for f_2.

<u>Inverse function theorem says :</u>

If you can solve this linear equation

and f_1, f_2 have continuous partial derivatives near (x_0, y_0),

then the inverse 'function'

(expressing (x, y) in terms of (z, w))

exists in a neighbourhood of (z_0, w_0).

$\boxed{\text{when } \det(\text{Jacobian}) \neq 0}$