Today:

Implicit function Theorem (12.8)

Recall: $F(x,y,z) = 0$ – defines a surface in 3-space ("implicit equation of a surface")

Normal vector to the tangent plane to this surface at (a,b,c):

~~note~~ we have 2 way of doing it:

* 1) $\bar{\nabla} F \big|_{(a,b,c)}$ is normal to the tangent plane

2) Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ using implicit differentiation,

implicitly "$z = z(x,y)$"

use $z - c = \frac{\partial z}{\partial x}\Big|_{(a,b)} (x-a) + \frac{\partial z}{\partial y}\Big|_{(a,b)} (y-b)$

normal: $\left\langle \frac{\partial z}{\partial x}\Big|_{(a,b)}, \frac{\partial z}{\partial y}\Big|_{(a,b)}, -1 \right\rangle$

What is the relation:

Recall: $\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$, $\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$

So our normal vector here is:

$$\left\langle -\frac{\partial F/\partial x}{\partial F/\partial z}, -\frac{\partial F/\partial y}{\partial F/\partial z}, -1 \right\rangle$$

$$= \left(-\frac{1}{\partial F/\partial z}\right) \cdot \bar{\nabla} F.$$

$\leadsto$ scale by $-\partial F/\partial z$

There is no difference, unless $\frac{\partial F}{\partial z} = 0$.

If this happens, then the way (1) works, the way (2) fails.

This means: if $\frac{\partial F}{\partial z} = 0$ then

$$\bar{\nabla} F = \langle --, --, 0 \rangle$$

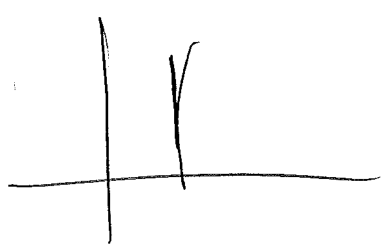
– parallel to the xy-plane.

We know: $\bar{\nabla} F$ is perpendicular to the tangent plane at our point (a, b, c).

This means, the tangent plane is vertical (~~the~~. parallel to the z-axis).

Now: recall in single variable

vertical $\iff$ y is not a diff. function of x.



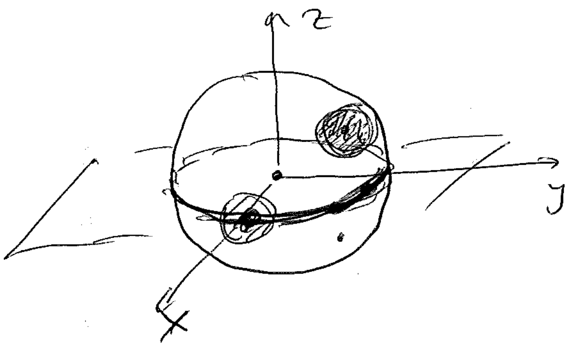
So our situation where $\bar{\nabla} F$ is parallel to the xy-plane indicates that we cannot think of z as a function of x, y in any neighbourhood of the point (a, b, c).

Example: $x^2+y^2+z^2=4$. $F(x,y,z)=x^2+y^2+z^2-4$.

where is $\frac{\partial F}{\partial z}=0$?

$(a,b,c)=(a,b,0)$ with $a^2+b^2=4$, then

$$\left.\frac{\partial F}{\partial z}\right|_{(a,b,c)}=0.$$



Imagine solving for z:

$$z=\pm\sqrt{4-(x^2+y^2)}$$

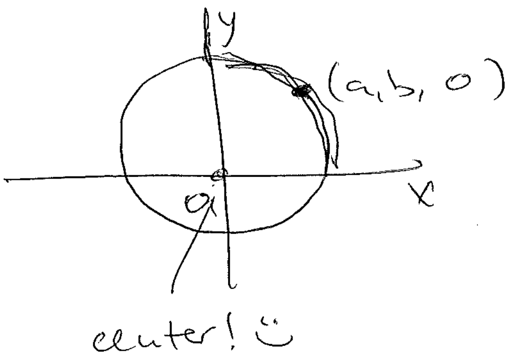
anywhere if $a^2+b^2<4$,
we have 2 solutions,
the point (a,b,c) either
has $c>0$ - upper hemisphere
or $c<0$ - lower hemisphere

Then there is a neighbourhood of (a,b,c) in which z is an honest function of (x,y).

But on the equator, No such neighbourhood!

Note: at points $(a,b,0)$ on the equator, actually x or y is an implicit function of the other 2 variables.

most of the time, both (when $a,b \neq 0$).

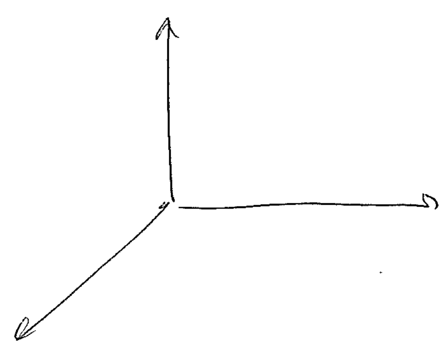


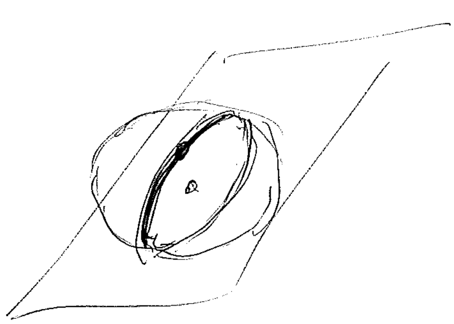
The point: $F(x,y,z)=0$ does not by itself tell you which variables to consider as independent variables.

Now: Suppose we had a "surface" defined by a system of equations?

ex: curve in $\mathbb{R}^3$ is typically defined by 2 equations:

Ex: $$\begin{cases} x^2+y^2+z^2=4 \\ 3x+z-y=0 \end{cases}$$





a 'big circle' on our sphere.

or: 2 equations in 4 variables typically define an 'honest' (2-dimensional) surface sitting in $\mathbb{R}^4$.

In our example of a circle, can still think of z as an implicit function of x, y

(or of x as an implicit function of y, z —).

Still can ask, what is $\frac{\partial z}{\partial x}$?

Could solve for z, y in terms of x.

Again: use implicit differentiation rather than solve. Then instead of solving a system of nonlinear equations, we will have to solve a system of linear equations.

So, take our system and differentiate with respect to x, thinking of both y and z as functions of x.

Note: here both z and y depend on x. We cannot think of y as an independent variable!

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2-4\right):$$

$$2x + 2y\frac{\partial y}{\partial x} + 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial}{\partial x}\left(3x+z-y\right):$$

$$3 + \frac{\partial z}{\partial x} - \frac{\partial y}{\partial x} = 0$$

Got:
$$\begin{cases} 2x + 2y\frac{\partial y}{\partial x} + 2z\frac{\partial z}{\partial x} = 0 \\ 3 + \frac{\partial z}{\partial x} - \frac{\partial y}{\partial x} = 0 \end{cases}$$

The number of independent variables = (number of all variables) − (number of equations)

Got 2 equations on $\frac{\partial y}{\partial x}$ and $\frac{\partial z}{\partial x}$.

And these equations are linear

Will continue: Recall ~~C~~ Cramer's rule!!