Lecture 2

Last time:
- Fixed coordinates in $\mathbb{R}^3$
- Defined $\mathbb{R}^n$.

1) Distances.

Mathematically, we are free to define
"distance "any way we like"
- will do it so it agrees with the familiar
  notion
  (Euclidean distance)

Try to get a formula for distance
between $A = (a_1, b_1, c_1)$ and
$B = (a_2, b_2, c_2)$ in terms of $(a_1, b_1, c_1)$
and $(a_2, b_2, c_2)$.

Recall: in $\mathbb{R}^1$

\[
\begin{array}{c}
\text{distance } |AB| = |b-a|
\end{array}
\]

in $\mathbb{R}^2$

\[
\begin{array}{c}
|AB| = \sqrt{(b_2-b_1)^2 + (a_2-a_1)^2}
\end{array}
\]
in $\mathbb{R}^3$

\[ B(a_2, b_2, c_2) \]

A

\( (a_1, b_1, c_1) \)

'Conjecture': \[ |AB| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2} \]

Have to prove it: has to agree with the notions of distance on every plane.

**Proof:**

\[ B = (a_2, b_2, c_2) \]

\[ \text{projection of } B \text{ to } xy\text{-plane} \]

\[ \text{xy-plane} \]

\[ B' \]

(Can assume \[ A = (0, 0, 0) \].)

\[ \text{Coordinates of } B': \]

\( (a_2, b_2, 0) \)

\[ |AB| = \sqrt{|AB'|^2 + |B'B|^2} \]

\[ |AB'|^2 = \left(\sqrt{(b_2 - b_1)^2 + (a_2 - a_1)^2}\right)^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2 \]

Using $\mathbb{R}^2$ distance formula

\[ |B'B|^2 = |c_2| \]
\[ |AB| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2} \]

assuming

A is in the xy-plane
so \( a_3 = 0 \)

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**Distance in \( \mathbb{R}^n \)**

\[ A = (a_1, a_2, \ldots, a_n) \quad B = (b_1, b_2, \ldots, b_n) \]

\[ |AB| = \sqrt{(a_1-b_1)^2 + \ldots + (a_n-b_n)^2} \]

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**Equations and inequalities in \( \mathbb{R}^3 \)**

1) **Equation of a sphere of radius \( R \)**

centered at \( A = (a, b, c) \):

Recall: sphere = set of points in \( \mathbb{R}^3 \)
whose distance from \( A = (a, b, c) \) equals \( R \)

\[ \{ P(x, y, z) \mid |AP| = R \} \]

\[ \{ (x, y, z) \mid (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \} \]

*equation of the sphere*
2) Planes in $\mathbb{R}^3$:

$(x, y, z) \in \mathbb{R}^3$

\[ ax + by + cz = d \] - general form of equation of a plane

(by analogy with line in $\mathbb{R}^2$:

\[ ax + by = c \]

(!) though it is analogous to a line in $\mathbb{R}^2$,

it defines a plane not a line in $\mathbb{R}^3$!

- reasonable that one equation in 3 variables will define a 2-dimensional object.

3) How to define a line?

Need 2 equations, e.g.

\[
\begin{align*}
ax + by + c_1z &= d_1, \\
ax_2 + by + c_2z &= d_2
\end{align*}
\] - intersection of two planes = line. (usually)
4) **Examples of equations**:

(a) \( x = y \)

Since \( z \) does not participate in the equation, for any point \( P = (x, y, 0) \), there is a whole vertical line through \( P \).

(b) \( z^2 = y \)

**Cylinder**: any surface consisting of parallel lines.

Plane curve + line gives a cylinder.
Inequalities:

\[ 3x + 5z \geq 1 \]

- half-space

\[ 3x + 5z = \frac{1}{5} \] = plane parallel to y-axis

first octant