Last time: $f: \mathbb{R}^n \to \mathbb{R}$

$$\vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$ – vector of directional derivatives.

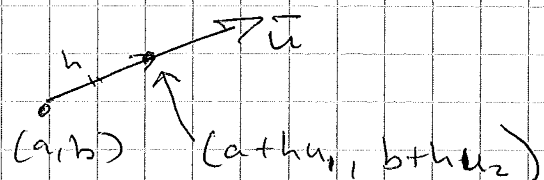
Today: $n = 2$ or $3$.
Talk about geometric meaning
(works for any $n$).

First: Directional derivatives.

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y)$.

$\bar{u}$ – unit vector
$\langle u_1, u_2 \rangle$

$$D_{\bar{u}} f\Big|_{(a,b)} = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$




If you want a directional derivative in a direction given by $\bar{v}$ = not unit vector,

then first compute $\bar{u} = \frac{\bar{v}}{|\bar{v}|}$ – unit.

Use this $\bar{u}$.

Reformulation:

$$\boxed{D_{\bar{u}} f\Big|_{(a,b)} = \bar{\nabla} f\Big|_{(a,b)} \cdot \bar{u}}$$

(when $f$ is differentiable at $(a,b)$).

$\cdot$ — dot product.

Why: by Chain rule:

Recall: example of a fly in the room,

$\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$ - fly's path

$T(x,y,z)$

$$\frac{dT(\bar{r}(t))}{dt} = \bar{\nabla} T \cdot \bar{v}$$

$\bar{v}$ = velocity of the fly = $\langle x'(t), y'(t), z'(t) \rangle$

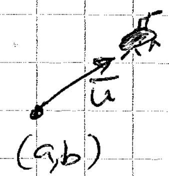
Now we are doing this in the plane;

let's have a "measurement bug":

always ~~cr~~ crawls with speed $= 1$.

Velocity of the bug $\bar{v}(t) = \langle x'(t), y'(t) \rangle$,

for any $t$, this is a unit vector.




imagine $f(x,y)$ is the temperature.

Let at $t=0$, our bug is at this point.

By def, $D_{\bar{u}} f(x,y)\Big|_{(a,b)}$ = rate of change of temp. in the direction $\bar{u}$,

$\|$ b/c the speed $=1$, time $=$ distance.

rate of change of temp. for the bug with respect to time

$\|$ by chain rule (see a few classes ago)

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \bar{\nabla} f\Big|_{(a,b)} \cdot \bar{v}$$

$\bar{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle\Big|_{t=0}$ = velocity of the bug.

The rule $\quad D_{\bar{u}} f = \bar{\nabla} f \cdot \bar{u}$ $\quad$ – works in any $\mathbb{R}^n$

(any $n$.)

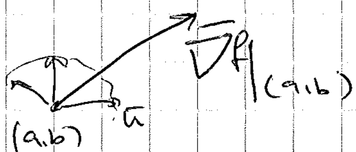
Geometric meaning:

$f(x,y)$ or $f(x,y,z)$ – given.

What happens as we vary $\bar{u}$?

Note: $\bar{\nabla} f\big|_{(a,b)}$ – fixed

$$\left|\nabla f \cdot \bar{u}\right| \le \|\bar{\nabla} f\| \cdot \overbrace{\|\bar{u}\|}^{\text{“}1}$$

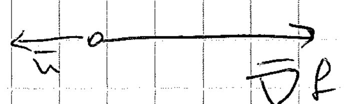



↑ length of vector.

We get: $\left|\text{Directional derivative}\right| \le$ length of the gradient.

And the fastest change occurs in the direction of the gradient, ← fastest increase

and opposite to it ← fastest decrease




direction of the fastest decrease of $f(x,y)$.

$$\bar{\nabla} f \cdot \bar{u} < 0$$

↑ as big negative as it can get.

Gradient points in the direction of the fastest increase of the function.

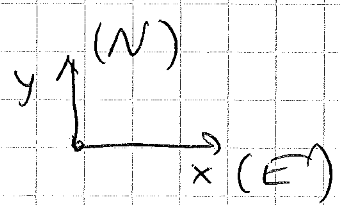
Caution "steepest ascent":

$z = f(x, y)$ ~ our mountain is the graph of this function.

North/south

east/west

altitude (above sea level).




"steepest ascent" = greatest increase of $z$ with respect to changes in $x, y$.

= the greatest $D_{\bar{u}} f(x, y)$

looking for $\bar{u}$.

So $\bar{u}$ has to point in the same direction as $\bar{\nabla} f$.

! The gradient $\bar{\nabla} f$ is a vector on the map.

When you go on the mountain in this direction, it means the projection of your velocity onto the $xy$-plane is parallel to $\bar{\nabla} f$.

• $\bar{\nabla} f \cdot \bar{u} = 0$ when $\bar{u} \perp \bar{\nabla} f$

$$\overset{\shortparallel}{D_{\bar{u}}} f = 0$$

so in 2 dimensions (for $f(x,y)$)

the gradient is perp. to level curves.

• For $f(x,y,z)$, $\bar{\nabla} f$ is perp. to level surfaces

Ex: $f(x,y,z) = 2x^2 + 3y^2 + 5z^2 - 1$

Level surfaces are ellipsoids.

We did this example:

$\bar{\nabla} f\big|_{(a,b,c)}$ is the normal vector to the tangent plane to this surface at $(a,b,c)$.

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* When we say "$\bar{\nabla} f$ points in the direction of steepest ascent" - we have to mean that our mountain is the surface $z = f(x,y)$.

$z = f(x,y)$

Let $F(\cancel{z}x,y,z) = f(x,y) - z$

$$\bar{\nabla} F = \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle}$$

perpendicular to the tangent plane to our 'mountain' $z = f(x,y)$ at the point $(a,b,c)$

But $\bar{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ – vector <u>on the map</u>

= projection of $\bar{\nabla} F$ onto $xy$-plane.

– gives you <u>compass direction</u> so that when you follow this direction you are going <u>up</u> along the steepest path.

Example 7 in 12.7 p. 724.

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<u>Read</u>: $\boxed{12.8}$