Recall: last time defined $Df$ - a differential operator for a transformation $f: \mathbb{R}^n \to \mathbb{R}^m$.

$(Df$ was a Jacobian matrix defining a linear transformation $f: \mathbb{R}^n \to \mathbb{R}^m)$

linear operator

Note: When $m = 1$ (our class)

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \text{ - an (1xn) Jacobian matrix.}$$

can think of it as a vector-

$$\nabla \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \text{gradient vector of } f$$

Def (informal) $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at the point $(a_1, ..., a_n)$ if its differential $Df$ approximates it well.
**Definition (formal):** \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is differentiable at \((a,b)\) if
\[
\lim_{(h,k) \to (0,0)} \frac{f(a+h, b+k) - f(a,b) - h f_x(a,b) - k f_y(a,b)}{\sqrt{h^2+k^2}} = 0
\]

**Discussion:**
Numerator in the limit is:
the difference between
\( f(a+h, b+k) \) and \( df + f(a,b) \)
\[
\begin{align*}
&\overset{\text{a point}}{\longrightarrow} \\
&\text{close to } (a,b) \\
&(b/k \quad (h,k) \text{ is close} \\
&\quad \text{to } (0,0))
\end{align*}
\]
We expect:
\[
\Delta f = f(a+h, b+k) - f(a,b) \approx df
\]
comes from
the linearization.
Our limit is  
\[
\lim_{(h,k) \to (0,0)} \frac{f - \text{(its linear approximation)}}{\text{distance from } (h,k) \text{ to } (0,0)}
\]

Compare this with one-variable:

\[
f(x) = f(a) + f'(a)(x-a) + \ldots
\]

Def of \(f'(a)\):

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)
\]

Let's see what happens if we take difference of \(f(x)\) and its linear approx. and divide by \(h\):

\[
f(a+h) - (f(a) + f'(a) \cdot h)
\]

\[
= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} - \lim_{h \to 0} \frac{f'(a) \cdot h}{h} = 0.
\]

When \(f'(a)\) exists
Digression: How to compare functions going to 0.

Want to have a scale of functions $f(x) \to 0$ as $x \to 0$ to measure "how fast" they go to 0.

**Definition:** $f(x) \to 0$ faster than $g(x)$ if
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = 0
\]

\[\overline{\text{if}}\]
\[
\lim_{x \to 0} g(x) = \infty \quad \text{DNE}
\]

\[
\frac{f(x)}{g(x)} \to 0 \quad \text{at the same speed as } g(x)
\]

\[
\text{if } \lim_{x \to 0} \frac{f(x)}{g(x)} = c \neq 0 \quad \text{exists}
\]

**Examples:** $x^2$ goes to 0 faster than $x$.

In general, $x^a$ goes to 0 faster than $x^b$ if and only if $a > b$.

'Scale': 1. $x, x^2, x^3, x^4, \ldots$
Look at approximations to \( f(x) \) near \( a \):

0) \( f(x) \approx f(a) \).

How good is this?

\[ f(x) - f(a) \to 0 \quad \text{as} \quad x \to 0 \]

(faster than \( x^0 \)).

Since \( f'(a) \) exists, in fact

\[ f(x) - f(a) \text{ goes to } 0 \text{ as fast as } (x-a) \]

if \( f'(a) \neq 0 \)

or even faster!

1) \( f(x) = (f(a) + f'(a)(x-a)) \)

How fast does this go to 0?

Faster than \( x-a \). — This is our earlier calculation.

In fact, if \( f''(a) \) exists, it actually goes to 0 as fast as \( (x-a)^2 \).

(Taylor approximations!)
Back to 2 variables:

Our definition says:

\[ f - \text{(linear approx.)} \] goes to 0 faster than distance from \( (x, y) \) to \( (a, b) \).

\[ f(x, y) \]

(Note: for comparison,

\[ f(x, y) \text{ is continuous at } (a, b) \]

\[ \lim_{(x, y) \to (a, b)} (f(x, y) - f(a, b)) = 0 \]

Discussion of the picture: (see the link to this post separately)

\[ f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & x \neq 0 \\ 0 & x = 0 \end{cases} \]

Both exist.

But \( f(x, y) \) is NOT differentiable at \( (0, 0) \) using our definition.

Each-trajectory fixed y is:

\[ \frac{df}{dx} \text{ gets steeper, approaches vertical.} \]

\[ \frac{df}{dx} \text{ is not continuous at } 0. \]
Theorem: If \( \frac{df}{dx} \) and \( \frac{df}{dy} \) exist and are continuous on a neighbourhood of \((a, b)\), then \( f \) is differentiable.

Proof: uses MVT (Mean Value Theorem) for functions of 2 variables, see 12.6.

Directional derivatives:

Def: Let \( \mathbf{u} = \langle u_1, u_2 \rangle \) be a unit vector \((u_1^2 + u_2^2 = 1)\).

\[
D_{\mathbf{u}} \left. \frac{df}{dx} \right|_{(a, b)} = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}
\]

directional derivative of \( f \) in the direction \( \mathbf{u} \) at \((a, b)\)

\[
= \nabla f \cdot \mathbf{u}
\] (When \( f \) is differentiable at \((a, b)\)).