Recall: last time defined Df – a differential operator

for a transformation $f: \mathbb{R}^n \to \mathbb{R}^m$

(Df was a Jacobian matrix defining

a $\underbrace{\text{linear } \underbrace{\text{transformation}}_{\text{operator}}}_{\text{linear operator}}$ $: \mathbb{R}^n \to \mathbb{R}^m$).

Note: When $m = 1$ (our class)

$$Df = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$$ – an $(1 \times n)$ – Jacobian matrix.

can think of it as a vector

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \text{gradient vector of } f$$

nabla

Def (informal) $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at the point $(a_1, \ldots, a_n)$

if its differential Df approximates it well. .

Def: (formal): $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - h f_x(a,b) - k f_y(a,b)}{\sqrt{h^2+k^2}} = 0$$

Discussion:

Numerator in the limit is:

the difference between

$f(a+h, b+k)$ and $df + f(a,b)$.

a point close to (a,b)

(b/c (h,k) is close to $(0,0)$)

$$df = f_x(a,b) \cdot \underbrace{dx}_{h} + f_y(a,b) \cdot \underbrace{dy}_{k}$$

We expect:

$$\Delta f = f(a+h, b+k) - f(a,b) \approx df$$

comes from the linearization.

Our limit is

$$\lim_{(h,k)\to(0,0)} \frac{f - (\text{its linear approximation at } (a+h, b+k))}{\text{distance from } (h,k) \text{ to } (0,0)}$$

Compare this with one-variable:

$$f(x) \approx f(a) + f'(a)(x-a) + \cdots$$

Def of $f'(a)$:

$$\boxed{\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)}$$

Let's see what happens if we take difference of $f(x)$ and its linear approx. and divide by h:

$$\lim_{h\to 0} \frac{f(a+h) - (f(a) + f'(a)\cdot h)}{h} =$$

$$= \underbrace{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}_{f'(a)} - \underbrace{\lim_{h\to 0} \frac{f'(a)\cdot \cancel{h}}{\cancel{h}}}_{f'(a)} = 0.$$

$\boxed{\text{when } f'(a) \text{ exists}}$

Digression: How to compare functions going to 0:

Want to have a scale of functions $\xrightarrow[x\to 0]{} 0$ to measure "how fast" they go to 0.

Def: $f(x) \to 0$ faster than $g(x)$ if $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

[assume: $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$]

$\Updownarrow$

$\lim_{x\to 0} \frac{g(x)}{f(x)} = \cancel{\infty}$ DNE

$f(x) \to 0$ at the same speed as $g(x)$ if $\lim_{x\to 0} \frac{f(x)}{g(x)} = c \neq 0$

exists.

Examples: x^2 goes to 0 faster than x.

in general, x^α goes to 0 faster than x^β if and only if $\alpha > \beta$.

'Scale': (1) $x, x^2, x^3, x^4, \ldots$

Look at approximations to $f(x)$ near a:

(assume for now f has 2 continuous derivatives: f', f'' exist, cont on a neighbourhood of a).

0) $f(x) \approx f(a)$.

How good is this?

$f(x) - f(a) \to 0$ as $x \to 0$.
(faster than x^0).

since $f'(a)$ exists, in fact

$f(x) - f(a)$ goes to 0 as fast as $(x-a)$

if $f'(a) \neq 0$

or even faster!

i) $f(x) - (f(a) + f'(a)(x-a))$

How fast does this go to 0?

Faster than $x-a$. — this is our earlier calculation

In fact, if $f''(a)$ exists, it actually goes to 0 as fast as $(x-a)^2$.

(Taylor approximations!)

Back to 2 variables:

Our definition says:

$f - $ (linear approx.) goes to 0 faster than distance from (x,y) to (a,b).

" $f(x,y)$

(**Note**: for comparison, $f(x,y)$ is <u>continuous</u> at (a,b) if

$$\lim_{(x,y)\to(a,b)} \left(f(x,y) - f(a,b)\right) = 0$$

Discussion of the picture: (see the link to this post separately)

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & x \neq 0 \\ 0 & x = 0 . \end{cases}$$

$\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)}$, $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)}$ both exist.

But $f(x,y)$ is <u>NOT</u> differentiable at $(0,0)$ using our definition.

Each trace with fixed y is:

← gets steeper, approaches vertical.

$\dfrac{\partial f}{\partial x}$ is not continuous at 0.

Theorem: If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on a neighbourhood of (a,b), then f is differentiable.

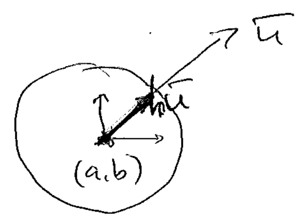
Proof: uses MVT (Mean Value Thm) for functions of 2 variables, see 12.6.

Directional derivatives:

Def: Let $\bar{u} = \langle u_1, u_2 \rangle$ be a unit vector ($u_1^2 + u_2^2 = 1$).

$$D_{\bar{u}} f \Big|_{(a,b)} = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

directional derivative of f in the direction $\bar{u}$ at (a,b)



$$= \boxed{\nabla f \cdot \bar{u}}$$ (when f is differentiable at (a,b)).