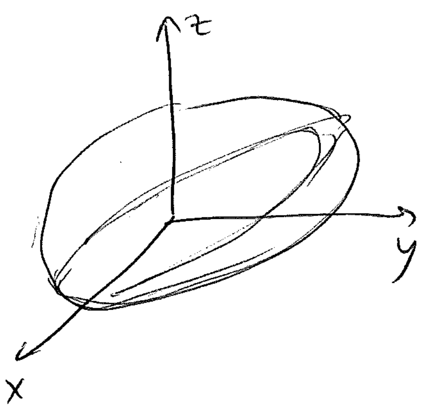
* Implicit differentiation :

Last time : $F(x,y,z) = 0$ - implicit equation of a surface

EX:



$$2x^2 + 3y^2 + 5z^2 = 1.$$

$$F(x,y,z) = 2x^2+3y^2+5z^2-1$$

* can consider z as an 'implicit function' of x, y

(could solve for z:

$$z = \pm \sqrt{\frac{1-2x^2-3y^2}{5}}$$

not a function of x, y: have to make a choice.

Given (a,b,c) on the ellipsoid, can check: if $c \geq 0$, use $z = +\sqrt{\ \cdots}$

if $c < 0$, use $z = -\sqrt{\ \cdots}$

To find tangent plane at (a,b,c), could differentiate z w.r.t. x, y, get:

$$z = \underset{z(a,b)}{\underset{\shortparallel}{c}} + \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(a,b)} (x-a) + \left.\frac{\partial z}{\partial y}\right|_{(a,b)} (y-b).$$

- equation of the tangent plane.)

Instead, use implicit differentiation:

Want to find $\left.\frac{\partial z}{\partial x}\right|_{(a,b)}$ and $\left.\frac{\partial z}{\partial y}\right|_{(a,b)}$

faster:

differentiate $F(x,y,z) = 0$

keeping in mind that "$z = z(x,y)$".

(implicit function)

$\frac{\partial}{\partial x}\left(F(x,y,z) = 0\right)$

(differentiate both sides):

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\boxed{\left.\frac{\partial z}{\partial x}\right|_{(a,b,c)} = \left.\frac{-\partial F/\partial x}{\partial F/\partial z}\right|_{(a,b,c)}}$$

Similarly, $\boxed{\left.\frac{\partial z}{\partial y}\right|_{(a,b,c)} = \left.\frac{-\partial F/\partial y}{\partial F/\partial z}\right|_{(a,b,c)}}$

Example: $F(x,y,z) = 2x^2 + 3y^2 + 5z^2 - 1$

$\frac{\partial F}{\partial x} = 4x$ $\qquad$ $\frac{\partial F}{\partial z} = 10z$.

$\frac{\partial F}{\partial y} = 6y$ $\qquad$ $\boxed{\frac{\partial z}{\partial x} = -\frac{4x}{10z}}$ $\qquad$ $\boxed{\frac{\partial z}{\partial y} = -\frac{6y}{10z}}$

ex: Let ~~[illegible]~~ $= (a, b, c) = \left(\frac{1}{10}, \frac{1}{5}, c\right)$

$$2 \cdot \frac{1}{100} + 3 \cdot \frac{1}{25} + 5c^2 = 1. \quad \boxed{c > 0.}$$

Tangent plane at this point is:

$$z = c + \left(-\frac{2}{5} \cdot \frac{1/10}{c}\right)\left(x - \frac{1}{10}\right) + \underbrace{\left(-\frac{3}{5}\right) \cdot \frac{1/5}{c}}_{\frac{\partial z}{\partial y}} \cdot \left(y - \frac{1}{5}\right)$$

Better form:

We got

$$z = c - \frac{\partial F/\partial x}{\partial F/\partial z}(x-a) - \frac{\partial F/\partial y}{\partial F/\partial z}(y-b)$$

Multiply by $\partial F/\partial z$.

Get:

$$\boxed{\frac{\partial F}{\partial z}(z-c) + \frac{\partial F}{\partial x}(x-a) + \frac{\partial F}{\partial y}(y-b) = 0}$$

Benefit: now ok to have a vertical tangent plane.

Note: Normal vector to this tangent plane is

$$\vec{n} = \underbrace{\left\langle \frac{\partial F}{\partial x}\Big|_{(a,b,c)}, \frac{\partial F}{\partial y}\Big|_{(a,b,c)}, \frac{\partial F}{\partial z}\Big|_{(a,b,c)} \right\rangle}_{\text{gradient of } F}$$

Recall: Last time:

Jacobian matrix:

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, meaning:

$f = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots x_n))$

Then

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If plug in $(a_1, \ldots, a_n)$ - a point from the domain,

get $D\big|_{(a_1 \ldots a_n)} f$ - matrix of numbers.
- linear transformation

(think linear algebra!).

We get:

$$\underbrace{\overset{\in \mathbb{R}^m}{f(x_1 \ldots x_n)} - \overset{\in \mathbb{R}^m}{f(a_1 \ldots a_n)}}_{\text{vector in } \mathbb{R}^m} \approx Df\big|_{(a_1 \ldots a_n)} \cdot \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}$$

(see next page for the relation between this formula and linearization)

Linearization: (Recall):

$f : \mathbb{R}^n \to \mathbb{R}$ - honest function

$$Df = \left[\frac{\partial f}{\partial x_1} \; - - \; \frac{\partial f}{\partial x_n} \right]$$

$(a_1 - - a_n)$ - a fixed point in the domain.

Linearization of $f(x_1 - x_n)$ at $(a_1 - a_n)$

is:

$$L(x_1 - - x_n) = f(a_1 - - a_n) + \left.\frac{\partial f}{\partial x_1}\right|_{(a_1 - a_n)} (x_1 - a_1)$$

$$+ \; \text{———} \; + \left.\frac{\partial f}{\partial x_n}\right|_{(a_1 - a_n)} (x_n - a_n)$$

$$= f(a_1 - a_n) + \underbrace{\left. Df \right|_{(a_1 - a_n)} \cdot \langle x_1 - a_1, \dots, x_n - a_n \rangle}_{\left[\frac{\partial f}{\partial x_1} \; - - \; \frac{\partial f}{\partial x_n} \right] \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}}$$

$$\underbrace{L(x_1, - , x_n) - f(a_1 - a_n)}_{\text{change in } f} = \left. Df \right|_{(a_1 - a_n)} \cdot \langle x_1 - a_1, - - x_n - a_n \rangle$$

Linear transformation Df is called the <u>differential</u> of $f(x_1 \ldots x_n)$

(it is the linear transformation whose matrix is the Jacobian matrix).

In the case $f = f(x_1 \ldots x_n)$ - a real-valued <u>function</u>

$(m = 1)$

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \ldots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

common to denote the vector

$$\begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix} \text{ by } \vec{dx} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

- <u>think of it as a vector of variables</u>.

Differential is a function of both:

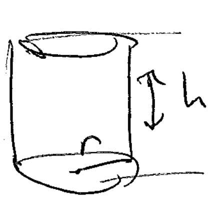
$x_1 \ldots x_n$ ← were called $(a_1 \ldots a_n)$

and $dx_1 \ldots dx_n$

<u>Example</u>: Using differentials to approximate functions:

volume of a cylinder : $V = \pi r^2 \cdot h$

$V(r, h)$



(Approximately)
How much metal was used to make this cylinder if its walls are 0.2 cm thick, bottom, top are 0.1 cm thick,

and $r = 5$ cm,

$h = 15$ cm.

$$\boxed{dV} = \underbrace{\left[2\pi r h, \ \pi r^2\right]}_{DV} \cdot \begin{bmatrix} dr \\ dh \end{bmatrix}$$

$$\frac{\partial V}{\partial r} = 2\pi r h$$

$$\frac{\partial V}{\partial h} = \pi r^2$$

$dV \approx$ the amount of metal used.

$\|$

$$\underbrace{2\pi \cdot 5 \cdot 15}_{\frac{\partial V}{\partial r}} \cdot \underbrace{0.2}_{dr} \text{ cm}^3 + \underbrace{\pi \cdot 5^2}_{\frac{\partial V}{\partial h}} \cdot \overbrace{(0.1 + 0.1)}^{dh} \text{ cm}^3$$

top bottom

$= \ldots$

Read: 12.6, 12.7
