- Written assignment extended till Monday Oct 26

No off. hr today.

Some off hrs on Friday -TBA.

Today:
- chain rule (general form)
- Implicit differentiation
- a word about PDE.

Recall: we had $T(x,y,z)$

plug in: $x(t)$, $y(t)$, $z(t)$

differentiate w.r.t. t.

2 ideas:
- single variable chain rule
- linear approx. of a fn of several variables

Today: $f(x,y,z)$

$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$

(in this: $n = 3$, $k = 2$

plug in 3 functions of 2 variables into a fn of 3 variables).

$$F(u,v) = f(x(u,v), y(u,v), z(u,v))$$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

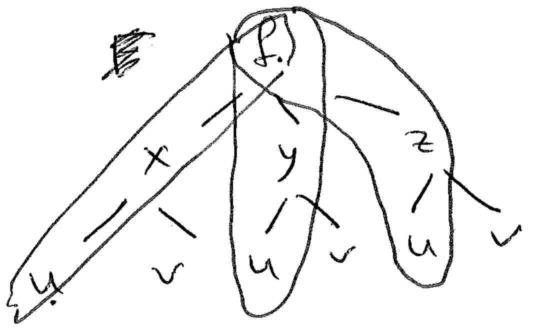
$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$$

This rule works because of the one we proved last class.

Last time: $f(x, y, z)$
$\quad x \overset{\parallel}{=} x(t) \quad y = y(t) \quad z = z(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

Ways to remember:



To compute $\frac{\partial f}{\partial u}$,
Sum over all paths that connect "f" with "u"

On a path, multiply the partials.

$$\begin{matrix} f \\ | \\ x \\ | \\ u \end{matrix} \longrightarrow \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u}$$

Example $f(x,y) = x^2 + \sin(y)x$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2x + \sin(y) \\ \frac{\partial f}{\partial y} &= x\cos(y) \end{aligned}}$$

$x(t, s, w) = e^t \cdot (s+w)$
$y(t, s, w) = \ln(s^2 - w) + t$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x + \sin y) \cdot e^t + x\cos y \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}\left(\ln(s^2 - w) + t\right) = \frac{1}{s^2 - w} \cdot 2s$$

$$= \Big(\underbrace{2e^t(s+w) + \sin(\ln(s^2-w)+t)}_{2x+\sin y}\Big)\cdot e^t$$

$$+ \underbrace{e^t(s+w)\cos(\ln(s^2-w)+t)}_{x\cos y} * \frac{2s}{s^2-w}$$

Better way to remember chain Rule :

Back to the case $f(x,y,z)$, $x = x(u,v)$, —

we can write

$$\underbrace{\left[\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right]}_{\substack{\cancel{2\times 1}\text{ - matrix.} \\ 1\times 2}} = \underbrace{\left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right]}_{\substack{\cancel{3\times 1}\text{ - matrix} \\ 1\times 3}} \underbrace{\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}}_{\substack{\cancel{2\times 3}\text{ - matrix} \\ 3\times 2}}$$

Generally : $\langle x(u,v), y(u,v), z(u,v) \rangle$

make a vector from our 3 functions.

Then we can think of it as a "function" from $\mathbb{R}^2$ to $\mathbb{R}^3$.

because takes 2 inputs: u, v.

In general, if we have

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

m functions of $x_1, \dots, x_n$

$f_1(x_1, \dots, x_n), \dots, f_m(x_1 \dots x_n)$

we can make a matrix of partial derivatives

$$Df = \begin{array}{c} \\ \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{array}\right] \\ \begin{array}{ccc} x_1 & \cdots & x_n \end{array} \end{array} \leftarrow \text{all partials of } f_1$$

Jacobian matrix

In our example with $f(x,y) = x^2 + x\sin(y)$,

$$f : \mathbb{R}^2 \to \mathbb{R}^1$$

so Jacobian matrix will have 1 row 2 columns:

$$\begin{bmatrix} 2x + \sin y, & x\cos y \end{bmatrix}$$

$f: \mathbb{R}^n \to \mathbb{R}^m \quad , \quad g: \mathbb{R}^m \to \mathbb{R}^k$

$$F = g(f(x_1, \ldots, x_n)) = \langle F_1, \ldots, F_k \rangle$$

$$= \langle F_1(x_1, \ldots, x_n), \ldots, F_k(x_1, \ldots, x_n) \rangle$$

<u>Recall</u>: single variable

$$\big(g(f(x))\big)' = g'(f(x)) \cdot f'(x).$$

<u>Now</u>: $$DF = Dg(f(x_1, \ldots, x_n)) \cdot Df$$

In our examples so far, we always had $k=1$.

This form of writing chain rule helps us not get confused:

1) compute Jacobian matrices separately
2) multiply them.

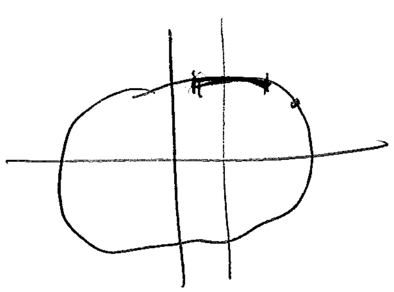
<u>Implicit differentiation</u> (12.8, first 1.5 pages).

Recall from calc. 1:

$f(x,y) = c$ ← this gives y as an <u>implicit</u> fn of x.

level curve of $f(x,y)$.

Task: find $\frac{dy}{dx}$.



Differentiate this w.r.t x, keeping in mind that y depends on x.

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial x}}_{= 1} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} = 0 = \frac{\partial}{\partial x}(c).$$

our unknown derivative (should write $\frac{dy}{dx}$).

get: $$\boxed{\frac{dy}{dx} = - \frac{\partial f/\partial x}{\partial f/\partial y}}$$

* Can do this also for

$F(x,y,z) = c$

z is implicit fn of x, y.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= - \frac{\partial F/\partial x}{\partial F/\partial z} \\ \frac{\partial z}{\partial y} &= - \frac{\partial F/\partial y}{\partial F/\partial z} \end{aligned}}$$

- derive this!