<u>Today</u>: 1) Higher derivatives
2) Chain Rule.

<u>Mixed partials are equal</u>:

(Clairaut's theorem)

"it does not matter in what order you do the differentiation" as long as $f(x_1 \ldots x_n)$ is 'smooth enough'.

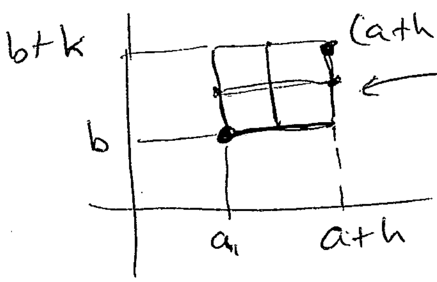
<u>ex</u>: <u>(the main case)</u>:

$f(x,y)$ - a function of 2 variables

- $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are <u>continuous</u> on a <u>neighbourhood</u> of (a,b)
- $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are cont. <u>at</u> (a,b)

Then $$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)}.$$

<u>Why this theorem works</u>:



h, k - small.

use Mean Value Thm on this rectangle.

(4 times)

(look at the proof: $\underline{\underline{12.4}}$).

<u>Example</u> , <u>Given</u> : $\begin{cases} \dfrac{\partial f}{\partial x} = x^2 - 3y + 1 \\ \dfrac{\partial f}{\partial y} = y^2 - 3x + \cos(y) \end{cases}$

Can you find f?

Using $\frac{\partial f}{\partial x}$, try to find $f(x,y)$:

integrate w.r.t. x, "keep y constant":
(with respect to)

unknown function

$$f(x,y) = \frac{x^3}{3} - 3y \cdot x + x + \cancel{x}\, h(y)$$

does not change w.r.t. x

it will help us get the right $\frac{\partial f}{\partial y}$.

Now take this, differentiate w.r.t. y:

$$\frac{\partial f}{\partial y} = 0 - \underline{\underline{3x}} + 0 + \frac{dh}{dy}$$

$$= y^2 \underline{\underline{-3x}} + \cos(y)$$

Then $\dfrac{dh}{dy} = y^2 + \cos(y)$

$$h(y) = \frac{y^3}{3} + \sin(y) + C$$

<u>Answer</u>: $f(x,y) = \dfrac{x^3}{3} - 3yx + x + \dfrac{y^3}{3} + \sin(y) + C$

What if we modified the example:

$$\begin{cases} \dfrac{\partial f}{\partial x} = x^2 - 3xy + 1 \\[2ex] \dfrac{\partial f}{\partial y} = y^2 - 3xy + \cos(y) \end{cases}$$

Would the same solution work?

Try: $f(x,y) = \dfrac{x^3}{3} - 3y \cdot \dfrac{x^2}{2} + x + g(y)$

Plug into the second equation:

$$\frac{\partial f}{\partial y} = \boxed{-3\frac{x^2}{2}} + \frac{dg}{dy} \underset{\text{want}}{=} y^2 \boxed{-3xy} + \cos(y)$$

Cannot find $g(y)$ s.t. $\dfrac{dg}{dy} = y^2 + \cos(y) - 3xy + 3$

this depends on x!

So the solution to this system of equations does not exist we could tell it won't exist, at least "smooth enough" $f(x,y)$ could not exist:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

In our example, $\underbrace{\frac{\partial}{\partial y}\left(x^2 - 3xy + 1\right)}_{= -3x} \neq$

$$\neq \underbrace{\frac{\partial}{\partial x}\left(y^2 - 3xy + \cos(y)\right)}_{= -3y}$$

Note: it's automatically 'smooth enough':

our $\frac{\partial f}{\partial x} = x^2 - 3xy + 1$ and $\frac{\partial f}{\partial y} = y^2 - 3xy + \cos(y)$ are continuous everywhere

and their derivatives are continuous here.

So: we really have no solution

Chain Rule (12.5)

Also: Read: 12.6
beginning of 12.8

Recall: $f(g(x))' = f'(g(x)) \cdot g'(x)$

Problem: there is a fly in the room, zooms around
with coordinates $(x(t), y(t), z(t))$ - function of time t

temperature in the room $T(x, y, z)$

What is the $\underbrace{\text{change of temp.}}$ that the fly is experiencing

"$\dfrac{dT}{dt}$" - derivative of temperature function the fly experiences w.r.t. time?

For the fly, temperature is a function of just one variable - time.

Answer:

$$\frac{dT^{fly}}{dt} = \frac{dT(x(t), y(t), z(t))}{dt}$$

$$= \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial T}{\partial z} \cdot \frac{dz}{dt}$$

Why this works (idea of proof).

Recall: linearization (linear approximation)

$$T(x,y,z) \approx T(x_0,y_0,z_0) + \left.\frac{\partial T}{\partial x}\right|_{(x_0,y_0,z_0)} \cdot (x-x_0) + \left.\frac{\partial T}{\partial y}\right|_{(x_0,y_0,z_0)} (y-y_0) + \left.\frac{\partial T}{\partial z}\right|_{(x_0,y_0,z_0)} (z-z_0)$$

at point (x_0,y_0,z_0); suppose the fly gets to that point at time $t=t_0$.

(x,y,z) is close to (x_0,y_0,z_0)

Let $\Delta T = T(x,y,z) - T(x_0,y_0,z_0)$ = change of temp.

$\Delta x = x - x_0$, ...

We have:

$$\Delta T \approx \left.\frac{\partial T}{\partial x}\right|_{(x_0,y_0,z_0)} \cdot \Delta x + \left.\frac{\partial T}{\partial y}\right|_{(x_0,y_0,z_0)} \cdot \Delta y + \left.\frac{\partial T}{\partial z}\right|_{(x_0,y_0,z_0)} \cdot \Delta z \qquad (*)$$

This knows nothing about the fly.

Look at the fly:

$$\Delta x \approx x'(t_0)\cdot\Delta t = \left.\frac{dx}{dt}\right|_{t=t_0}\cdot\Delta t$$

$$\Delta y \approx \left.\frac{dy}{dt}\right|_{t=t_0}\cdot\Delta t$$

$$\Delta z \approx \left.\frac{dz}{dt}\right|_{t=t_0}\cdot\Delta t$$

Plug this into $(*)$

Get: $$\Delta T \approx \left(\left.\frac{\partial T}{\partial x}\right|_{(x_0,y_0,z_0)}\cdot\left.\frac{dx}{dt}\right|_{t=t_0} + \left.\frac{\partial T}{\partial y}\right|_{(x_0,y_0,z_0)}\cdot\left.\frac{dy}{dt}\right|_{t=t_0} + \left.\frac{\partial T}{\partial z}\right|_{(x_0,y_0,z_0)}\cdot\left.\frac{dz}{dt}\right|_{t=t_0}\right)\Delta t$$

Now look at $\lim_{\Delta t\to 0}\frac{\Delta T}{\Delta t} = \left.\frac{dT}{dt}\right|_{t=t_0}$.

Why this is not rigorous: have to worry about how good our approximations were