Today: 12.3, 12.4 - Partial derivatives.

~~Recall~~ Recall:

Def $f(x,y)$ - a function of 2 variables.

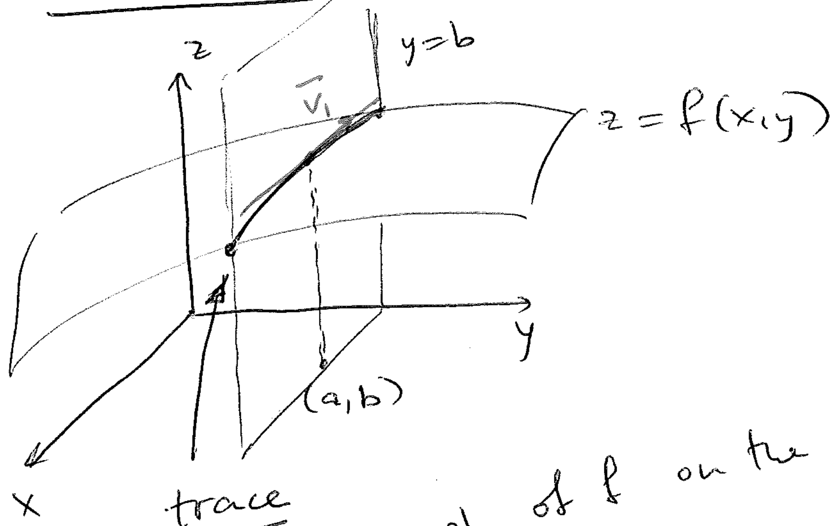
$$f_x(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

x-partial derivative of f at the point (a,b)

the book uses $f_1(a,b)$. I will not.

$$\frac{\partial f}{\partial y}\Big|_{(a,b)} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Geometric meaning

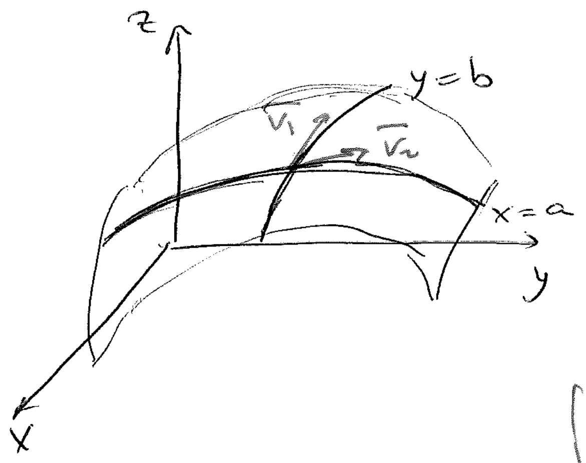



trace of the graph of f on the plane y=b.

$\frac{\partial f}{\partial x}\Big|_{(a,b)}$ is the slope of the tangent line to the trace on y=b plane

in the sense of "$\frac{\Delta z}{\Delta x}$"

Fix x=a, $\frac{\partial f}{\partial y}\Big|_{(a,b)}$ is the slope of the tangent line to the trace on the plane x=a.




The tangent vectors to the traces <u>span</u> the tangent plane to the graph at the point $(a, b, f(a,b))$

- if <u>the function is smooth enough</u>
- will come back to it, see weird examples.

Assume $f$ is "smooth enough"

How <u>to find the equation of the tangent plane to the graph of $f$</u>?

<u>2 ways</u>:

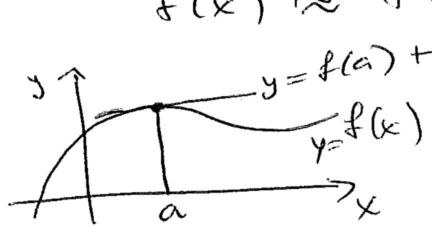
1. "$f$ is smooth enough" should mean that we can write a reasonable linear approximation to $f(x,y)$ in a neighbourhood of $(a,b)$:

$$f(x,y) \approx \boxed{f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b)}$$

linear approximation to $f(x,y)$ near $(a,b)$.
Its graph is a plane. This is the tangent plane at $(a,b)$

Recall: single variable: linear approx.

$$f(x) \approx \overbrace{f(a) + f'(a)(x-a)} + \underbrace{\frac{f''(a)}{2}(x-a)^2 + \ldots}_{\text{higher-order terms.}}$$



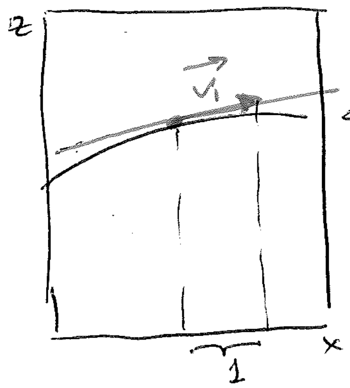
## ② <u>Second way of thinking about the tangent plane</u> :

Write an equation of the plane containing the two tangent lines to the traces on $x=a$, $y=b$.

<u>Normal</u>: cross product of the two tangent vectors to these traces.

<u>The tangent vectors to the traces</u> :

the trace on $y=b$ plane:




$y=b$
← our tangent line, slope $= \dfrac{\partial f}{\partial x}\Big|_{(a,b)}$.
← our trace

$$\bar{v}_1 = \Big\langle 1, 0, \underbrace{\frac{\partial f}{\partial x}\Big|_{(a,b)}}_{\text{the slope}} \Big\rangle$$

In the same way, the direction vector of the tangent line to the trace on $x=a$ plane is:

$$\bar{V}_2 = \Big\langle 0, 1, \frac{\partial f}{\partial y}\Big|_{(a,b)} \Big\rangle$$

$$\bar{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x}\Big|_{(a,b)} \\ 0 & 1 & \frac{\partial f}{\partial y}\Big|_{(a,b)} \end{vmatrix} = \Big\langle -\frac{\partial f}{\partial x}\Big|_{(a,b)}, -\frac{\partial f}{\partial y}\Big|_{(a,b)},$$

<u>Note</u>: if we use this vector and the point $(a, b, f(a,b))$, we get exactly the same equation as before!

<u>Note</u>: The first way works in any dimension, but this only works for a function of 2 variables.

Def: For $f(x_1, \ldots, x_n)$: linear approx. at the point $(a_1, \ldots, a_n)$ is

$$f(x_1, \ldots, x_n) \approx f(a_1, \ldots, a_n) + \left.\frac{\partial f}{\partial x_1}\right|_{(a_1, \ldots, a_n)} (x_1 - a_1)$$

$$+ \left.\frac{\partial f}{\partial x_2}\right|_{(a_1, \ldots, a_n)} (x_2 - a_2) + \cdots$$

$$+ \left.\frac{\partial f}{\partial x_n}\right|_{(a_1, \ldots, a_n)} (x_n - a_n)$$

One can prove that this is the best possible of all linear approximations.

Read: 12.4.