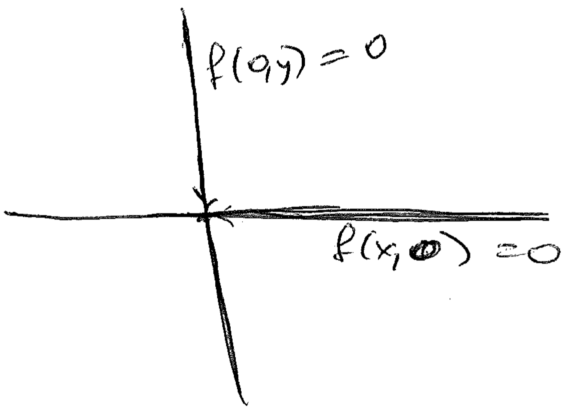
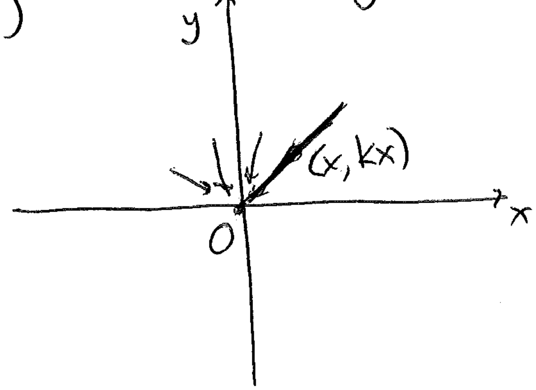
Recall: we discussed how a function can have a limit 'along the x-axis'

1) and 'along y-axis', but not have a limit as $f(x,y)$.



2) On every line through the origin $f(x, kx)$ has a limit as $x \to 0$



but $f(x,y)$ does not have a limit.

Example $\dfrac{x^2 y}{x^4 + y^2} = f(x,y)$

Plug in $y = kx$ $(k \neq 0)$

Get: $f(x, kx) = \dfrac{kx^3}{x^4 + (kx)^2} = \dfrac{kx^3}{x^4 + k^2x^2}$

$$= \frac{k\cancel{x^2}\cdot x}{\cancel{x^2}(x^2+k^2)} \qquad \downarrow x \to 0$$

$$= \frac{kx}{x^2 + k^2} \xrightarrow[x \to 0]{} 0$$

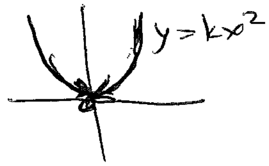
When $k=0$, then $f(x,0) = 0$.

So, <u>along every line</u>, $f(x,y) \to 0$ as $(x,y)\to(0,0)$

The <u>limit</u> (in the sense of 2 variables)

Does NOT EXIST!

<u>Try $y=kx^2$:</u>



$$f(x,kx^2) = \frac{x^2\cdot kx^2}{x^4+k^2x^4}$$

$$= \frac{k}{1+k^2} \quad \text{- depends on } k!$$

<u>Claim</u>: if $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$ exists

(L has to be 0 in this case)

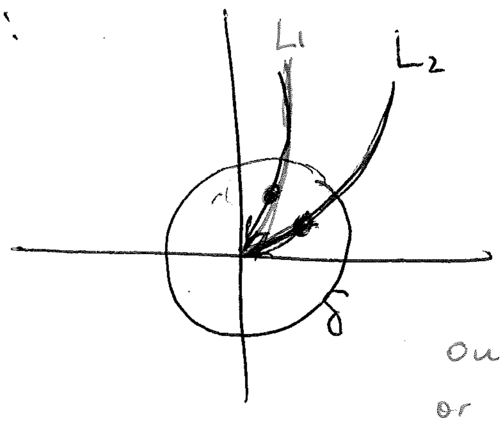
then along <u>any curve</u> $y=g(x)$ passing through the origin

$$\lim_{x\to 0} f(x,g(x)) = L \quad - \boxed{hw}$$

Summary: along lines, $f(x,y) \to 0$ as $x \to 0$
but along parabolas, $f(x,y) \to$ different numbers depending on which parabola you choose.

Claim: Then limit does not exist.

Why:

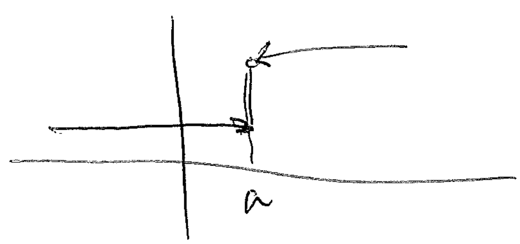


Take $\varepsilon = \frac{|L_1 - L_2|}{2}$.

Want δ: $|f(x,y) - 0| < \varepsilon$ when $(x,y) \in B_\delta(0,0)$.
on red parabola, it is close to L_1
or black parabola, close to L_2.

Upshot: there is no way to define the value $f(0,0)$ to make it continuous at $(0,0)$.

(This is like jump discontinuity:



)

Contours: $\leftarrow$ for this function (looked at a contourplot, and a graph, from wolframalpha)

$$\frac{x^2 y}{x^4+y^2} = k$$

$$x^2 y = k(x^4+y^2)$$

Example: $\dfrac{x^4 y}{x^4+y^2} \longrightarrow 0$

Proof: Given ε, want δ, s.t. when $\sqrt{x^2+y^2} < \delta$, then

$$\left| \frac{x^4 y}{x^4+y^2} \right| < \varepsilon.$$

Scratchwork: want to estimate $\left| \dfrac{x^4 y}{x^4+y^2} \right|$

'from above':

note: $|\underbrace{x^4}_{\geq 0} + \underbrace{y^2}_{\geq 0}| \geq \begin{matrix} |x^4| \leftarrow \boxed{\text{use this}} \\ |y^2| \end{matrix}$ — can use whichever is more convenient.

Get: $\left| \dfrac{x^4 y}{x^4+y^2} \right| \leq \left| \dfrac{x^4 y}{x^4} \right| = |y|.$

Can take $\delta = \varepsilon/2$ — not really needed, but doesn't hurt :)

$\delta = \min\left(\frac{\varepsilon}{2}, 1\right)$ — maybe not needed.

Take our $\delta = \varepsilon/2$

If $\sqrt{x^2+y^2} < \delta$ then $|y| < \delta$.

Then $\left| f(x,y) \right| = \left| \frac{x^4 y}{x^4+y^2} \right| \leq |y| < \delta = \varepsilon/2 < \varepsilon$.

Done!