Recall: last time we proved with $(\varepsilon - \delta)$ that

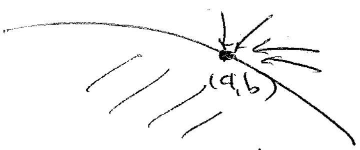
$$\lim_{(x,y)\to(2,0)} \frac{\sqrt{x^2+y^2-4}}{3x+y} = 0.$$

How to find such limits without ε-δ.

- Continuity:

Def: $f(x,y)$ is continuous at (a,b)

if $f(x,y)$ is defined in a neighbourhood of (a,b) ~~~~ if (a,b) is a boundary point of the domain of f.



and $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

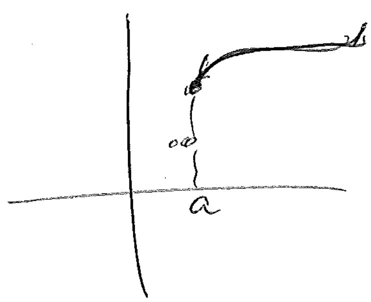
and equals $f(a,b)$.

(**Recall:** $f(x)$ on $\mathbb{R}$

$f(x)$ is cont. at $x=a$ if $\lim_{x\to a} f(x) = f(a)$)

one-sided:

(on the right).



$\lim_{x\to a+} f(x) = f(a)$

Our function $f(x,y) = \dfrac{\sqrt{x^2+y^2-4}}{3x+y}$

is continuous at $(2,0)$

– that's what we proved using our ε-δ argument.

- Generally, how to prove a function is continuous at a point:

  - you prove once and for all that:

    x^n, y^n, $\exp(x)$, $\exp(y)$, $\log(x)$ - -

    $\sin(x)$ ——

    same function

    are continuous (know from calc. 1) of <u>one</u> variable

  - check that if you think of them as fns of 2 variables ($f(x,y) = y^n$) they are still continuous on the whole domain.

- prove that if f, g are continuous at (a,b), then:

  $f(x,y) + g(x,y)$ is continuous

  $f(x,y) \cdot g(x,y)$ is continuous

  $\dfrac{f(x,y)}{g(x,y)}$ is continuous if $g(a,b) \neq 0$.

  $f(x,y) = F(g(x,y))$ is cont. at (a,b)

  function of a single variable continuous at $g(a,b)$

  <u>ex</u>: $\sin(\underbrace{x^2+y^2}_{= g(x,y)})$

  F

In our example, use these properties:

- x^2+y^2-4 is a polynomial, so it is continuous

- $\sqrt{t}$ is continuous. $(t \geq 0)$.

Then $\sqrt{x^2+y^2-4}$ is cont.

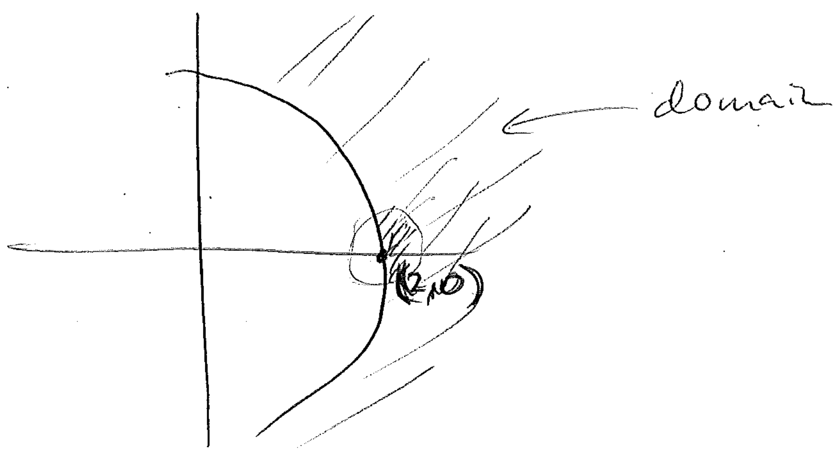
The denominator $3x+y$ is cont. and

$3 \cdot 2 + 0 \neq 0$, so can use

the rule about the quotient.

Then $f(x,y) = \dfrac{\sqrt{x^2+y^2-4}}{3x+y}$ is cont. at $(2,0)$.

Then $\lim\limits_{(x,y)\to(2,0)} f(x,y) = f(2,0) = 0$



<u>Why do ε-δ:</u>

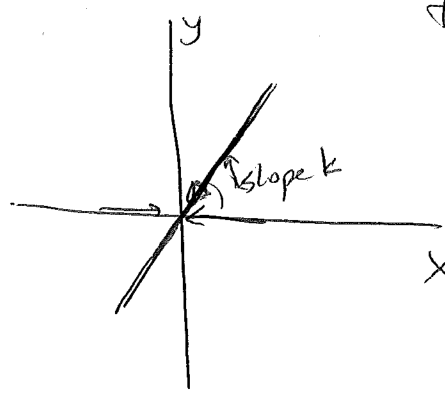
* There's no other way to establish the continuity of the 'basic' functions - powers, exp, etc.

* Sometimes cannot use any rules about continuity,

<u>ex:</u> $f(x,y) = \dfrac{xy}{x^2+y^2}$ - at $(a,b)=(0,0)$.

Example: $f(x,y) = \dfrac{xy}{x^2+y^2}$

Caution!
rules from calc 1
about $\frac{0}{0}$
do not really work!



try to approach 0
along x-axis:

$g(x) = f(x,0)$
$g(x) \equiv 0$!
$h(y) = f(0,y)$
$h(y)$ is also identically 0!

$\lim_{x\to 0} g(x) = \lim_{y\to 0} h(y) = 0.$

Try to approach $(0,0)$

along $y = kx$

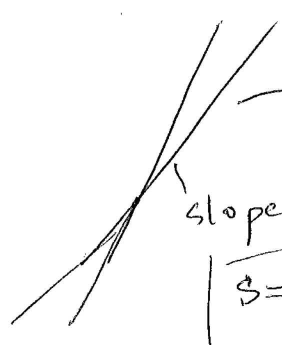
Let $u(x) = f(x, kx)$

$$\lim_{x\to 0} u(x) = \lim_{x\to 0} \frac{x\cdot kx}{x^2+(kx)^2} = \lim_{x\to 0} \frac{k\cancel{x^2}}{(1+k^2)\cancel{x^2}}$$

$$= \lim_{x\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2} \neq 0$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? — NO.

~~Example~~: In the contour plot,



these lines are where
$f(x,y) = s$ ← equally spaced.

slope k
$\boxed{s = \dfrac{k}{1+k^2}}$

4

More about contour plot:

$\boxed{\text{Ex 5}}$ $s = f(x,y)$ $\Rightarrow$ find all x, y.

$$s = \frac{xy}{x^2+y^2}$$

$$\boxed{s(x^2+y^2) - xy = 0}$$ — what does this look like?

1) Note: the set of (x,y) satisfying this equation contains the line $y = kx$

where $\frac{k}{1+k^2} = s$.

(if you can find such k).

ex: look at $s = 1$.

$k = 1 + k^2$

$k^2 - k + 1 = 0$

No real roots.

Then this line doesn't work!

Take $s = \frac{1}{5}$: $k = \frac{1}{5}(1+k^2)$

$1 + k^2 - 5k = 0$

Find 2 roots, this gives 2 lines.

What happens when the line doesn't work out?

Two options: 1) s is not in the range of $f(x,y)$

2) The level curve is not a pair of lines.

Which is the case?? — homework.

What happens if you knew:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along any}\\ \text{straight line}}} f(x,y) = A$$

$$\parallel$$

$$\lim_{x\to 0} f(x,kx) = A.$$

Is it true then that

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ exists? (and equal } A\text{).}$$