Extra off. hrs

Tomorrow 4:30 - 6 pm

Fr: 9:30 - 10:30 am.

Last time: Defined $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

- saw an example where can tell from contour plot that the limit does not exist
- an indication (not proof) at a "good point" that the limit probably exists.

Note: contour plot cannot prove that there is a limit.

- Example: $f(x,y) = \dfrac{\sqrt{x^2+y^2-4}}{3x+y}$

$\lim_{(x,y)\to(2,0)} f(x,y) = \frac{0}{6} = 0$ ← got it by plugging in $(x,y)=(2,0)$.

↑ make a guess

Proof: (using ε/δ):

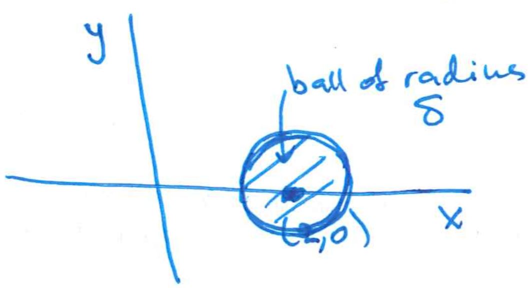
Need to show: $\forall \varepsilon > 0 \;\; \exists \delta$ s.t.

when $\sqrt{(x-2)^2+y^2} < \delta$ then we have

↑ distance from (x,y) to $(2,0)$.

$\left| \dfrac{\sqrt{x^2+y^2-4}}{3x+y} - 0 \right| < \varepsilon$

↑ our L.



Scratch work:

can control δ. goal: make this work

want to prove:

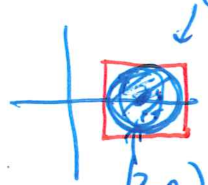
$$\left|\frac{\sqrt{x^2+y^2-4}}{3x+y}\right| < \varepsilon$$

$$\wedge\!\backslash$$

$$\sqrt{x^2+y^2-4}$$

remember: (x,y) is close to $(2,0)$ then $3x+y$ is close to 6. at least, it's going to be ≥ 1.

have: $(x,y) \in B_\delta(2,0)$



$(2,0)$

$$\sqrt{(x-2)^2+y^2} < \delta$$

means: (implies):

$$\begin{cases} |x-2| < \delta \\ \text{and} \\ |y| < \delta \end{cases}$$

rewrite these inequalities:

$$\boxed{\begin{array}{c} 2-\delta \leq x \leq 2+\delta \\ -\delta \leq y \leq \delta \end{array}}$$

have: got an estimate for x,y.

Plug it into the function:

Want:

$$\sqrt{x^2+y^2-4} < \varepsilon$$

$$x^2+y^2-4 < \varepsilon^2$$

Estimate x^2+y^2-4:

$$4-4\delta+\delta^2 \leq x^2 \leq 4+4\delta+\delta^2$$

$$0 \leq y^2 \leq \delta^2$$

$$\boxed{-4\delta+\delta^2+0 \leq x^2+y^2-4 \leq 4\delta+\delta^2+\delta^2}$$

$$|x^2+y^2-4| \leq 4\delta+2\delta^2$$

Want: $2\delta^2+4\delta < \varepsilon^2$

(Recall: ε is given, need such δ).

Take $\delta = \min\left(\frac{\varepsilon^2}{6}, \frac{1}{2}\right)$

(Replace $2\delta^2$ with 2δ)

$\boxed{\text{want } \delta < 1!}$

Making the proof:

We need to show that given ε, there is δ

s.t. when $\sqrt{(x-2)^2+y^2} < \delta$,

we have $\left| \dfrac{\sqrt{x^2+y^2-4}}{3x+y} \right| < \varepsilon$. $(*)$

Let $\varepsilon > 0$.

Take $\delta = \min\left(\frac{\varepsilon^2}{6}, \frac{1}{2}\right)$ this is needed as 'protection' from silly large ε.

Now have to prove that for any (x,y) s.t. $\sqrt{(x-2)^2+y^2} <$ our δ $= \min\left(\frac{1}{2}, \frac{\varepsilon^2}{6}\right)$

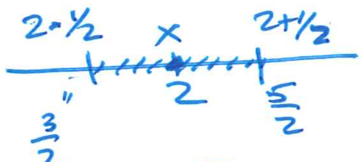
$(*)$ holds.

Proving: take (x,y).

Get: $|x-2| < \frac{\varepsilon^2}{6}$

$|y| < \frac{\varepsilon^2}{6}$. (and also $|x-2| < \frac{1}{2}$ and $|y| < \frac{1}{2}$).

Since $|x-2| < \frac{1}{2}$, $x > \frac{3}{2}$, $|y| < \frac{1}{2} \Rightarrow y > -\frac{1}{2}$



Then $3x+y > 3\cdot\frac{3}{2} - \frac{1}{2} > 1$

comes from y

Then $\dfrac{\sqrt{x^2+y^2-4}}{3x+y} < \sqrt{x^2+y^2-4}$.

Now let us estimate $\sqrt{x^2+y^2-4}$.

We know: (put in the work from "scratchpaper").

$$|x^2+y^2-4| \le 4\delta + 2\delta^2$$

$$\le 4\cdot\frac{\varepsilon^2}{6} + 2\cdot\left(\frac{\varepsilon^2}{6}\right)^2$$

(also $\le 4\cdot\frac{\varepsilon^2}{6} + 2\cdot\frac{1}{4}$ if $\frac{\varepsilon^2}{6} \ge 1$)

if $\frac{\varepsilon^2}{6} < 1$, get

$$2\cdot\left(\frac{\varepsilon^2}{6}\right)^2 < 2\cdot\frac{\varepsilon^2}{6}$$

$$\le 4\cdot\frac{\varepsilon^2}{6} + 2\cdot\frac{\varepsilon^2}{6} = \varepsilon^2$$

– this is <u>what we needed</u> to prove.