Written assignment 4. Due Wednesday November 25.
Review of gradients, Jacobians, Chain Rule, and the Implicit Function Theorem.

(1) Suppose it is given that the direction of the fastest increase of a function $f(x, y)$ at the origin is $\langle 1, 2\rangle$. Find a unit vector $\mathbf{u}$ that is tangent to the level curve of $f(x, y)$ that passes through the origin.

Solution. The direction of the fastest increase of a function f at a point is the direction of the gradient of f at that point. Thus, the gradient of $f(x, y)$ at the origin has the same direction as the vector $\langle 1, 2\rangle$. We also know that the gradient is orthogonal to the level curve passing through our point. Thus, the unit vectors tangent to the level curve passing though the origin are $\pm\langle -2/\sqrt{5}, 1/\sqrt{5}\rangle$.

(2) (From an old final exam) The shape of the hill is given by $z = 1000 - 0.1x^2 - y^2$. Assume that the x-axis is pointing East, and the y-axis is pointing North, and all distances are in metres.

(a) What is the direction of the steepest ascent at the point $(0, 20, 600)$? (The answer should be in terms of directions of the compass).
Solution. The compass direction of the steepest ascent at the given point is the direction of the gradient of $f(x, y) = 1000 - 0.1x^2 - y^2$ at $(0, 20)$. Evaluate this gradient. We have: $\nabla f = \langle -0.2x, -2y\rangle$, at $(0, 20)$ we get $\nabla f|_{(0,20)} = \langle 0, -40\rangle$, which points strictly South.

(b) What is the slope of the hill in the direction from (a)?
Solution. The slope is the rate of change of height in this direction, i.e. the directional derivative $D_{\mathbf{u}}f$ at $(0, 20)$ where $\mathbf{u}$ is the unit vector pointing South, i.e. $\mathbf{u} = \langle 0, -1\rangle$. We get:

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle 0, -40\rangle \cdot \langle 0, -1\rangle = 40.$$

This means that if you are trying to walk on this hill in the South direction, you ascend 40m per horizontal metre. Which actually shows that this "hill" is much more suitable for rock climbing than bike riding... (in reality something with this slope certainly appears as a completely vertical cliff). Oops.

(c) If you ride a bicycle on this hill in the direction of the steepest descent at 5 m/s, what is the rate of change of your altitude (with respect to time) as you pass through the point $(0, 20, 600)$?
Solution. Ignoring the issue with realism noted above, if you somehow were riding a bike off this cliff at 5 m/s along the path of the steepest descent (which is North), then your speed would have had two components: the horizontal component $\mathbf{v}_{xy}$ and the vertical component $\mathbf{v}_z$. (This part is similar to the faulty webwork problem discussed earlier). We need to find $|\mathbf{v}_z|$. We have: $|\mathbf{v}_z| = 40|\mathbf{v}_{xy}|$, because $\mathbf{v}$ has to be tangent to the path, and so the ratio of $|\mathbf{v}_z|$ and $|\mathbf{v}_{xy}|$ has to be the slope we found above. And also, $|\mathbf{v}_{xy}|^2 + |\mathbf{v}_z|^2 = 25$. Thus we get:

$1601|\mathbf{v}_{xy}|^2 = 25$, so

$$|\mathbf{v}_{xy}| = \frac{5}{\sqrt{1601}}, \quad \text{and } |\mathbf{v}_z| = \frac{200}{\sqrt{1601}}.$$

(3) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(r, \theta, z) = (x, y, z)$, where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$ (the *cylindrical coordinates* transformation, which we will use for computing integrals later). Find its Jacobian matrix (it should be a 3×3-matrix).

Solution. This has appeared in the last lecture. The answer is

$$\begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(4) Let $g(x, y, z) = (x^2 + y^2, x/y, 3z)$. Let $f(r, \theta, z)$ be the transformation from the previous problem. Find the Jacobian matrix of the transformation $g(f(r, \theta, z))$ in two ways: directly by finding the formula for this composition of transformations, and also by multiplying the Jacobian matrices of f and of g (i.e., by using the Chain Rule). (Hint: it should be diagonal).

Solution. By the Chain rule, the Jacobian matrix of the combined transformation is the product of their Jacobian matrices: $D(g \circ f) = DgDf$. The Jacobian matrix of g is:

$$\begin{bmatrix} 2x & 2y & 0 \\ 1/y & -x/y^2 & 0 \\ 0 & 0 & 3. \end{bmatrix}.$$

Combining this with the result from the previous problem, we get:

$$\begin{bmatrix} 2x & 2y & 0 \\ 1/y & -x/y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now since $(x, y, z) = f(r, \theta, z)$, we substitute the corresponding expressions for x, y, z, and get:

$$\begin{bmatrix} 2r\cos(\theta) & 2r\sin(\theta) & 0 \\ \frac{1}{r\sin(\theta)} & -\frac{\cos(\theta)}{r\sin^2(\theta)} & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2r & 0 & 0 \\ 0 & -1 - \frac{\cos^2(\theta)}{\sin^2(\theta)} & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Note: there is a much simpler solution in this case, by evaluation the composite map directly: note that $x^2 + y^2 = r^2$, and $x/y = \cot(\theta)$, and so $g(f(r, \theta, z)) = (r^2, \cot(\theta), 3z)$, and so its Jacobian matrix should be diagonal with the diagonal entries being the derivatives of r^2, $\cot(\theta)$, and $3z$, which is exactly the matrix we computed above. Note, however, that in general you cannot expect such simple formulas, and so you need to know how to use Chain rule and understand every step of the above solution.

(5) (Problem 14 on p.744) The equations $F(x,y,z) = 0$ and $G(x,y,z) = 0$ together can define any two of the variables x, y and z as functions of the remaining variable. Show that

$$\frac{dx}{dy}\frac{dy}{dz}\frac{dz}{dx} = 1.$$

Solution. Consider our system of equations

$$\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$$

and differentiate both equations with respect to x, keeping in mind that this system defines y and z as implicit functions of x, so every time we see y or z, we need to use chain rule. We get:

$$\begin{cases} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0 \\ \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} + \frac{\partial G}{\partial z}\frac{dz}{dx} = 0. \end{cases}$$

We got a system of *linear* equations, where $\frac{dy}{dx}$ and $\frac{dz}{dx}$ are unknowns. We can solve it:

$$\begin{cases} \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = -\frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial y}\frac{dy}{dx} + \frac{\partial G}{\partial z}\frac{dz}{dx} = -\frac{\partial G}{\partial x}, \end{cases} \quad \text{so} \quad \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial F}{\partial x} \\ -\frac{\partial G}{\partial x} \end{bmatrix}$$

Note that the matrix whose inverse appears is the *Jacobian matrix of this system of equations*: the partial derivatives of F and G with respect to the variables y and z which we have decided are *dependent* variables. Let's use the notation F_x instead of $\frac{\partial F}{\partial x}$ to make the solution simpler, and explicitly invert the Jacobian matrix. We get:

$$\frac{dy}{dx} = -\frac{F_x G_z - F_z G_x}{F_y G_z - F_z G_y}; \quad \frac{dz}{dx} = -\frac{F_y G_x - F_x G_y}{F_y G_z - F_z G_y}.$$

Before we proceed with the solution, three remarks:

(a) In the book, there is the notation $\frac{\partial(F,G)}{\partial(x,z)}$ to denote the determinant of a matrix or partial derivatives of the functions F and G with respect to the variables x, z. Then our answer would be written as:

$$\frac{dy}{dx} = -\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}},$$

which is precisely the kind of ratio that appears in the Implicit Function theorem.

(b) Since here y and z are implicit functions of a single variable, we use the notations $\frac{dy}{dx}$, $\frac{dz}{dx}$ rather than $\frac{\partial y}{\partial x}$.

(c) Unlike the situation in 2 dimensions where x, y are defined as implicit functions of each other by a single equation $F(x,y) = 0$, and where we have $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, here this relation does not hold, because all three variables x, y, z are "interdependent", and the second equation $G(x,y,z)$ participates in the derivatives, so all we get is the relation we are trying to prove in this problem, relating all three derivatives $\frac{dx}{dy}$, $\frac{dy}{dz}$, $\frac{dz}{dx}$.

Anyway, now we can proceed with the solution. In the same way, we get

$$\frac{dy}{dz} = -\frac{\frac{\partial(F,G)}{\partial(z,x)}}{\frac{\partial(F,G)}{\partial(y,x)}} = -\frac{F_xG_z - F_zG_x}{F_xG_y - F_yG_x},$$

$$\frac{dx}{dy} = -\frac{\frac{\partial(F,G)}{\partial(y,z)}}{\frac{\partial(F,G)}{\partial(x,z)}} = -\frac{F_yG_z - F_zG_y}{F_xG_z - F_zG_x}.$$

Multiplying all these together, we get -1.

(6) (modified Problem 15 on p.744) The equations

$$\begin{cases} x = u^3 - uv \\ y = 3uv + 2v^2 \end{cases}$$

define u and v implicitly as functions of x and y near the point P where $(u, v, x, y) = (-1, 2, 1, 2)$.

(a) Let $F(x, y, u, v) = u^3 - uv - x$, and $G(x, y, u, v) = 3uv + 2v^2 - y$ (the functions defining the two equations above). Write down the Jacobian matrix (differentiating the equations with respect to the variables u and v which we have decided to consider "dependent"): $\begin{bmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{bmatrix}$.

Answer:

$$\begin{bmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{bmatrix} = \begin{bmatrix} 3u^2 & -u \\ 3v & 3u + 4v \end{bmatrix}.$$

(b) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ at P (hint: use the answer form (a): you should see the inverse of this matrix).

Solution: We could differentiate our system of equations implicitly with respect to x, and every time we see u and v, think of them as implicit functions of x and y (the same method as in the previous problem and as what we did in class). We would get the system of linear equations on $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, and the matrix of this system is the Jacobian matrix we computed in (a):

$$\begin{bmatrix} 3u^2 & -u \\ 3v & 3u + 4v \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(Make sure that you know how we got this – do the differentiation!) Then

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 3u^2 & -u \\ 3v & 3u + 4v \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Plugging in P (whose (u, v)-coordinates are $(-1, 2)$, we get:

$$\begin{bmatrix} \frac{\partial u}{\partial x}|_P \\ \frac{\partial v}{\partial x}|_P \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -1 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/9 \\ -2/3 \end{bmatrix}.$$

Similarly for the y-partials:

$$\begin{bmatrix} \frac{\partial u}{\partial y}|_P \\ \frac{\partial v}{\partial y}|_P \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -1 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}.$$

(c) Find the approximate value of u when $x = 1.02$ and $y = 1.97$.
Solution. We are using the fact that we are near the point P, so $u_0 = -1$, $v_0 = 2$ (the values of u, v at P). Using the usual formula for the linearization of a function, we have

$$u \sim u_0 + \frac{\partial u}{\partial x}|_P \Delta x + \frac{\partial u}{\partial y}|_P \Delta y = -1 + \frac{5}{9} \cdot 0.02 - \frac{19}{\cdot}(-0.03).$$

(7) (modified Problem 16 on p.744) The equations

$$\begin{cases} u + v = x^2 + y^2 \\ u - v = x^2 - 2xy^2 \end{cases}$$

define x and y implicitly as functions of u and v for values of (x, y) near $(1, 2)$, and values of (u, v) near $(-1, 6)$.

(a) Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ at $(u, v) = (-1, 6)$.
Hint: write down the Jacobian matrix, differentiating the equations with respect to the dependent variables x and y in this case, as in the previous problem.
Solution. This problem is exactly the same as the previous one, except here the variables that we want to consider dependent are called x, y, and also the "independent" variables u, v are a bit more involved in the function, affecting the right-hand side of the linear system we need to solve. We have (make sure you understand how to get here!)

$$\begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -4xy \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Plugging in our point (where $(x, y) = (1, 2)$), we get:

$$\begin{bmatrix} \frac{\partial x}{\partial u}|_P \\ \frac{\partial y}{\partial u}|_P \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & -8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You should be able to finish it from here.

(b) If $z = \ln(y^2 - x^2)$, find $\frac{\partial z}{\partial u}$ at $(u, v) = (-1, 6)$.
Hint: use chain rule.
Solution. Again, as in the previous problem, once we found the partials of our implicit function, we can use them the way we usually use partials. One thing to note is that when $(u, v) = (-1, 6)$, we have $(x, y) = (1, 2)$ – this is the point on whose neighbourhood we have interpreted x and y as implicit functions of u and v.
Here we have, by chain rule,

$$\frac{\partial z}{\partial u} = -\frac{2x}{y^2 - x^2}\frac{\partial x}{\partial u} + \frac{2y}{y^2 - x^2}\frac{\partial y}{\partial u}.$$

Now just plug in $(x, y, u, v) = (1, 2, -1, 6)$ and the partials $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$ from (a).