Written assignment 4. Due Wednesday November 25.
Review of gradients, Jacobians, Chain Rule, and the Implicit Function Theorem.

(1) Suppose it is given that the direction of the steepest increase of a function $f(x, y)$ at the origin is $\langle 1, 2\rangle$. Find a unit vector $\mathbf{u}$ that is tangent to the level curve of $f(x, y)$ that passes through the origin.

(2) (From an old final exam) The shape of the hill is given by $z = 1000 - 0.1x^2 - y^2$. Assume that the x-axis is pointing East, and the y-axis is pointing North, and all distances are in metres.
   (a) What is the direction of the steepest ascent at the point $(0, 20, 600)$? (The answer should be in terms of directions of the compass).
   (b) What is the slope of the hill in the direction from (a)?
   (c) If you ride a bicycle on this hill in the direction of the steepest descent at 5 m/s, what is the rate of change of your altitude (with respect to time) as you pass through the point $(0, 20, 600)$?

(3) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(r, \theta, z) = (x, y, z)$, where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$ (the *cylindrical coordinates* transformation, which we will use for computing integrals later). Find its Jacobian matrix (it should be a 3×3-matrix).

(4) Let $g(x, y, z) = (x^2 + y^2, x/y, 3z)$. Let $f(r, \theta, z)$ be the transformation from the previous problem. Find the Jacobian matrix of the transformation $g(f(r, \theta, z))$ in two ways: directly by finding the formula for this composition of transformations, and also by multiplying the Jacobian matrices of f and of g (i.e., by using the Chain Rule). (Hint: it should be diagonal).

(5) (Problem 14 on p.744) The equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ together can define any two of the variables x, y and z as functions of the remaining variable. Show that

$$\frac{dx}{dy}\frac{dy}{dz}\frac{dz}{dx} = 1.$$

(6) (modified Problem 15 on p.744) The equations

$$\begin{cases} x = u^3 - uv \\ y = 3uv + 2v^2 \end{cases}$$

define u and v implicitly as functions of x and y near the point P where $(u, v, x, y) = (-1, 2, 1, 2)$.
   (a) Let $F(x, y, u, v) = u^3 - uv - x$, and $G(x, y, u, v) = 3uv + 2v^2 - y$ (the functions defining the two equations above). Write down the Jacobian matrix (differentiating the equations with respect to the dependent variables u and v): $\begin{bmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{bmatrix}$.
   (b) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ at P (hint: use the answer form (a): you should see the inverse of this matrix).

(c) Find the approximate value of u when $x = 1.02$ and $y = 1.97$.

(7) (modified Problem 16 on p.744) The equations

$$\begin{cases} u + v = x^2 + y^2 \\ u - v = x^2 - 2xy^2 \end{cases}$$

define x and y implicitly as functions of u and v for values of (x, y) near $(1, 2)$, and values of (u, v) near $(-1, 6)$.

(a) Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ at $(u, v) = (-1, 6)$.
Hint: write down the Jacobian matrix, differentiating the equations with respect to the dependent variables x and y in this case, as in the previous problem.

(b) If $z = \ln(y^2 - x^2)$, find $\frac{\partial z}{\partial u}$ at $(u, v) = (-1, 6)$.
Hint: use chain rule.