Written assignment 3. Due Wednesday October 21. Solutions.

(1) Using ϵ-δ definition, prove that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

Solution. Given $\epsilon > 0$, we need to find $\delta > 0$ such that when $\sqrt{x^2+y^2} < \delta$, we have

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \epsilon.$$

First of all, note that $x^2 + y^2 \geq y^2$, so $\sqrt{x^2+y^2} \geq \sqrt{y^2} = |y|$. Then

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{xy}{|y|} \right| = |x|.$$

Then we can take $\delta = \epsilon$. Indeed, suppose (x, y) is any point such that $\sqrt{x^2+y^2} < \delta$. Then in particular $|x| < \delta$, and then by the inequality we just proved above, we have

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq |x| < \delta = \epsilon,$$

so we have shown that for our point (x, y), $|f(x, y)| < \epsilon$; since (x, y) was an arbitrary point satisfying $\sqrt{x^2+y^2} < \delta$, this completes the proof.

(2) Using ϵ-δ definition, prove that $f(x, y) = x^2y$ is a continuous function on $\mathbb{R}^2$.

Solution. We need to show that for any point $(a, b) \in \mathbb{R}^2$, our function is continuous at (a, b). This means, that for any given $\epsilon > 0$, there exists δ (which can depend on ϵ, a, and b), such that for any point (x, y) in the ball of radius δ centred at (a, b), we have $|f(x, y) - f(a, b)| < \epsilon$. Concretely, this means: we need to find $\delta > 0$, such that the inequality $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ implies the inequality $|x^2y - a^2b| < \epsilon$.

In order to find such δ, we note that it would be very convenient to rewrite the expression $x^2y - a^2b$ in such a way that we would see the differences $(x - a)$ and $(y - b)$ in it (because then we can make these less than δ, estimate the remaining terms, and find the right δ). So, we do some algebra:

$$\begin{aligned} x^2y - a^2b &= x^2y - a^2y + a^2y - a^2b \\ &= (x^2 - a^2)y + a^2(y - b) = (x - a)(x + a)y + a^2(y - b). \end{aligned}$$

Using triangle inequality, we get:

$$|x^2y - a^2b| \leq |(x - a)(x + a)y| + |a^2(y - b)|.$$

Next, note that when x is close to a (say, $|x-a| < 1$), then $|x+a| \leq |x|+|a| \leq (|a| + 1) + |a| = 2|a| + 1$. Similarly, if $|y - b| < 1$, then $|y| < |b| + 1$. So, let us make sure that whatever δ we choose in the end, it should be less than 1. Then for any (x, y) inside the disc of radius δ around (a, b), we will have:

$$|x^2y - a^2b| \leq |(x - a)(x + a)y| + |a^2(y - b)| \leq |x - a|(2|a| + 1)(|b| + 1) + |y - b||a|^2.$$

Now, let us take

$$\delta = \min\left(1, \frac{\epsilon}{(2|a|+1)(|b|+1)+|a|^2}\right).$$

(Note that 1 appears there because of the above discussion: for our estimates to work, we need $\delta \le 1$).

Now, finally, we can put it all together: suppose

$$\sqrt{(x-a)^2+(y-b)^2} < \delta.$$

Then by the above estimates, we have:

$$|x^2y-a^2b| \le |x-a|(2|a|+1)(|b|+1)+|y-b||a|^2 < \delta\left((2|a|+1)(|b|+1)+|a|^2\right) = \epsilon,$$

and the proof is completed.

(3) Using the properties of continuous functions (you do not have to do an ϵ-δ proof), prove that the function defined by

$$f(x,y) = \begin{cases} (x^2+1)\frac{\sin(x^2+y^2)}{x^2+y^2} & (x,y)\ne(0,0) \\ 1 & (x,y)=(0,0) \end{cases}$$

is continuous at the origin.

Solution. Let $g(x,y) = x^2+1$, and let

$$h(r) = \begin{cases} \frac{\sin(r)}{r} & r\ne 0 \\ 1 & r=0 \end{cases}.$$

Then $f(x,y) = g(x,y)h(x^2+y^2)$. We know from Calculus 1 that $h(r)$ is a continuous function at $r=0$ (note that it is a function of a single variable!). Then $h(x^2+y^2)$ is continuous as composition of continuous functions. The function $g(x,y) = x^2+1$ is continuous as well (x is a continuous function of (x,y); the product of continuous functions is continuous; the sum of a continuous function x^2 and the constant function 1 is continuous). Then, $f(x,y)$ is continuous as a product of two continuous functions.

(4) (The "claim" from class on October 7):
Suppose $\lim_{(x,y)\to(0,0)} f(x,y) = L$ exists. Let $g(x)$ be any continuous function, such that $\lim_{x\to 0} g(x) = 0$. Prove that then the limit of $f(x,y)$ along the curve $y = g(x)$ (as x approaches 0) exists and equals L. In other words, prove that

$$\lim_{x\to 0} f(x, g(x)) = L.$$

Hint: the proof is very similar to (and simpler than) the proof of continuity of the composite function that we did in class on October 9.

Solution. We need to show that given $\epsilon > 0$, there exists δ such that if $|x| < \delta$, then $|f(x,g(x)) - L| < \epsilon$.

To find such δ, we 'unwind' the expression $f(x,g(x))$. We are given that $f(x,y)$ is a continuous function at the origin. This means, in particular, that for our given ϵ, there exists a value $\delta_f > 0$, such that when $\sqrt{x^2+y^2} < \delta_f$, we have $|f(x,y) - L| < \epsilon$. Let us compare this with what we want to prove: we want our δ to be such that when $|x| < \delta$, then $|f(x,g(x)) - L| < \epsilon$. This means, if we could only find a δ such that when $|x| < \delta$, then the point $(x, g(x))$ satisfies the condition $\sqrt{x^2+g(x)^2} < \delta_f$, then we would be done! Now we use the continuity of the function $g(x)$ at the origin. Let the value $\delta_f/\sqrt{2}$ play the role of "ϵ". Since $g(x)$ is a continuous function with $g(0) = 0$, we get that there exists δ_g such that when $|x| < \delta_g$, then $|g(x)| < \delta_f/\sqrt{2}$. Finally, take

$$\delta = \min\left(\frac{\delta_f}{\sqrt{2}}, \delta_g\right).$$

Let us prove that this δ "works". We need to prove: if $|x| < \delta$, then $|f(x, g(x)) - L| < \epsilon$. Suppose $|x| < \delta$. By definition of δ_g, we have that $|g(x)| < \delta_f/\sqrt{2}$. We also have that $|x| < \delta \leq \delta_f/\sqrt{2}$. Then

$$x^2 + g(x)^2 < \delta_f^2/2 + \delta_f^2/2 = \delta_f^2.$$

Then by definition of δ_f, we have that $|f(x, g(x)) - L| < \epsilon$, and the proof is completed.

(5) (Bonus question): Let $f(x, y)$ be a continuous function, and let r be a real number. Prove that the set

$$S = \{(x, y) \mid f(x, y) < r\}$$

is open.

Hint: Use the definition of an open set, and then the definition of a continuous function.

Solution. The solution is similar to the previous one. Let $(a, b) \in S$. We need to show that (a, b) is an interior point of S, which by definition means that there exists $\delta > 0$ such that the whole disc of radius δ centred at (a, b) is contained in S. By the definition of the set S, this means we need to find such δ that for any (x, y) satisfying $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have $f(x, y) < r$.

Since $(a, b) \in S$, we know that $f(a, b) < r$. Let $\epsilon = \frac{r - f(a,b)}{2}$; then it is a positive number. Since $f(x, y)$ is a continuos function, we know that for this value of ϵ, there exists $\delta > 0$ such that when $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have $|f(x, y) - f(a, b)| < \epsilon = \frac{r - f(a,b)}{2}$. But this implies that $f(x, y) < r$, and therefore $(x, y) \in S$, and the proof is completed.