Written assignment 3. Due Wednesday October 21.

Problems:

(1) Using ϵ-δ definition, prove that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

(2) Using ϵ-δ definition, prove that $f(x,y) = x^2y$ is a continuous function on $\mathbb{R}^2$.

(3) Using the properties of continuous functions (you do not have to do an ϵ-δ proof), prove that the function defined by

$$f(x,y) = \begin{cases} (x^2+1)^{\frac{\sin(x^2+y^2)}{x^2+y^2}} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

(4) (The "claim" from class on October 7):
Suppose $\lim_{(x,y)\to(0,0)} f(x,y) = L$ exists. Let $g(x)$ be any continuous function, such that $\lim_{x\to 0} g(x) = 0$. Prove that then the limit of $f(x,y)$ along the curve $y = g(x)$ (as x approaches 0) exists and equals L. In other words, prove that

$$\lim_{x\to 0} f(x, g(x)) = L.$$

Hint: the proof is very similar to (and simpler than) the proof of continuity of the composite function that we did in class on October 9.

(5) (Bonus question): Let $f(x,y)$ be a continuous function, and let r be a real number. Prove that the set

$$S = \{(x,y) \mid f(x,y) < r\}$$

is open.

Hint: Use the definition of an open set, and then the definition of a continuous function.