Written assignment 3. Due Wednesday October 21.

Problems:

1. Using $\epsilon$-$\delta$ definition, prove that
   \[
   \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.
   \]

2. Using $\epsilon$-$\delta$ definition, prove that $f(x, y) = x^2y$ is a continuous function on $\mathbb{R}^2$.

3. Using the properties of continuous functions (you do not have to do an $\epsilon$-$\delta$ proof), prove that the function defined by
   \[
   f(x, y) = \begin{cases} \frac{(x^2 + 1)\sin(x^2 + y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}
   \]
   is continuous at the origin.

4. (The "claim" from class on October 7):
   Suppose \( \lim_{(x,y) \to (0,0)} f(x,y) = L \) exists. Let \( g(x) \) be any continuous function, such that \( \lim_{x \to 0} g(x) = 0 \). Prove that then the limit of \( f(x, y) \) along the curve \( y = g(x) \) (as \( x \) approaches 0) exists and equals \( L \). In other words, prove that
   \[
   \lim_{x \to 0} f(x, g(x)) = L.
   \]
   *Hint: the proof is very similar to (and simpler than) the proof of continuity of the composite function that we did in class on October 9.*

5. (Bonus question): Let \( f(x, y) \) be a continuous function, and let \( r \) be a real number. Prove that the set
   \[
   S = \{ (x, y) \mid f(x, y) < r \}
   \]
   is open.
   *Hint: Use the definition of an open set, and then the definition of a continuous function.*