Written assignment 2. Due Wednesday September 30. Solutions.

Let S be a set in $\mathbb{R}^n$, as in Section 10.1. Also as in 10.1, we say that a set is *closed* if its complement S^c is open.

Recall also that the set S is called *open* if every point of this set has a neighbourhood (that is, an open ball centred at that point), which is contained in S. This is the same as saying that every point of S is an interior point.

Let ∂S denote the set of the boundary points of S.

Problems:

(1) Using these definitions, prove that S is a closed set if and only if $S \supseteq \partial S$.

Proof. First, observe the simple fact: the boundary of S is the same as the boundary of the complement of S (this can be written as $\partial S = \partial S^c$). This is so by definition of a boundary point (check and make sure you understand this statement).

Now, let us prove the required statement. We need to prove two statements:

(a) If S is closed, then $S \supseteq \partial S$, and

(b) If $S \supseteq \partial S$, then S is closed.

Proof of (1): We are given that S is closed, which, by definition of a closed set, means that its complement S^c is open.

Now we need to show that the set ∂S is contained in S, which means: if a point b belongs to ∂S, then b belongs to S. Take a point b in ∂S. We want to prove that b is in S, which is equivalent to proving that b *cannot* be contained in S^c. So, let us prove that a boundary point of S cannot be contained in S^c. Remember, we are given that S is closed, which means that its complement S^c is open. So if b belonged to S^c, then there would have been a whole neighbourhood of b contained entirely in S^c (by definition of an open set). But this contradicts the definition of a boundary point. Then b cannot be in S^c, and therefore it has to be in S, and the statement $\partial S \subseteq S$ is proved.

Proof of (2): We are given that $\partial S \subseteq S$, and we need to prove that S is closed, or equivalently, that S^c is open. A point a of S^c has only two options: it can be an interior point of S^c, or a boundary point of S^c. But if it was a boundary point of S^c it would have been a boundary point of S also; and we know that no boundary points of S belong to S^c since S contains them all. Thus, a cannot be a boundary point, and therefore has to be an interior point. So we proved that every point of S^c is interior to S^c, so S^c is open, and therefore S is closed.

(2) Is the complement of the x-axis in $\mathbb{R}^2$ an open subset? (Include proof).

Solution. Yes, it is open. Let S be the complement of the x-axis. Then $S = \{(x, y) \mid y \neq 0\}$.

We need to show that for any point $(x_0, y_0) \in S$, there exists a neighbourhood $B_r(x_0, y_0)$ of some radius r, such that $B_r(x_0, y_0) \subset S$. Let us take $r = |y_0|/2$. Now we want to show that if a point (x, y) belongs to the disc of radius r centred at (x_0, y_0), then $(x, y) \in S$. This means, we need to show that $y \neq 0$. Since $(x, y) \in B_r(x_0, y_0)$, we have $|y - y_0| < r$, which means, $y \in (y_0 - r, y_0 + r)$. We have chosen $r = |y_0|/2$. If $y_0 > 0$, then $y_0 - r = y_0/2$, and we have $y > y_0/2 > 0$. If $y_0 < 0$, then $r = -y_0/2$, and we have $y < y_0 + r = y_0/2 < 0$. In any case we see that $y \neq 0$, and therefore $(x, y) \in S$, and the proof is completed.

(3) (Problem 11 from 10.4) Give a condition on the position vectors of four points in $\mathbb{R}^3$ that guarantees that these four points lie in the same plane. Prove that your condition is necessary and sufficient.

Solution. Let $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_C$, and $\mathbf{r}_D$ be the position vectors of our four points A, B, C and D, respectively.

There exists a plane containing all four points if and only if the vectors $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$ lie in the same plane. These vectors can be expressed as: $\overline{AB} = \mathbf{r}_B - \mathbf{r}_A$, $\overline{AC} = \mathbf{r}_C - \mathbf{r}_A$,$\overline{AD} = \mathbf{r}_D - \mathbf{r}_A$.

Now, there are two cases: either the vectors $\overline{AB}$ and $\overline{AC}$ are parallel or not. Consider the case they are not parallel first. Then they define a plane. Then the condition is that the third vector, $\overline{AD}$, lies in this plane, which is equivalent to saying that it is perpendicular to the normal vector of this plane, which, in turn, is equivalent to:

$$\overline{AD} \cdot (\overline{AB} \times \overline{AC}) = 0.$$

Now note that if $\overline{AB}$ and $\overline{AC}$ were parallel, this condition would have been satisfied for any D, because then the cross product $\overline{AB} \times \overline{AC}$ would be zero; and also for any point D there would exist a plane containing A, B, C and D. So no additional condition is required to cover the second case.

Thus, we get that the condition (written in terms of the position vectors) is:

$$(\mathbf{r}_D - \mathbf{r}_A) \cdot ((\mathbf{r}_B - \mathbf{r}_A) \times (\mathbf{r}_C - \mathbf{r}_A)) = 0.$$

Remarks. 1. Of course you could use any other point (meaning, B, C, or D) as the "base point" for making your vectors. You would obtain an equivalent condition.

2. You could also say that for these points to be in the same plane, you need the normal to the plane (ABC) to be parallel to the normal of the plane (BCD) (again, some other choices of planes are possible here). This would lead to a seemingly different condition

$$((\mathbf{r}_C - \mathbf{r}_A) \times (\mathbf{r}_B - \mathbf{r}_A)) \times ((\mathbf{r}_D - \mathbf{r}_B) \times (\mathbf{r}_C - \mathbf{r}_B)) = 0.$$

(Note that here the parentheses are very important because cross product is not associative). In fact, this condition is equivalent to the one we derived first (so this is also a correct solution). It is a slightly tricky question to check their equivalence algebraically. If you want to do that (just out of curiosity), use Problems 23 and 25 on p.587 as hints.