Department of Mathematics

University of British Columbia

MATH 226 Final Exam

December 4, 2013, 12:00PM - 2:30PM

Family Name: ____________________________ Initials: ______

I.D. Number: ________________ Signature: ____________________________

CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.

JUSTIFY ALL OF YOUR ANSWERS (except as otherwise specified).

THERE ARE 8 PROBLEMS ON THIS EXAM.

$\log(x)$ **MEANS THE NATURAL LOGARITHM OF** x**.**

Question	**Mark**	**Out of**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		80

1.

(a) Find an equation of the tangent plane to the surface $2x^2 + 3y^2 + 4z^3 = 9$ at the point $(1, 1, 1)$.

(b) Find an equation of the tangent plane to the surface $z = 2x^2 - y^3$ at the point $(1, 1, 1)$.

(c) Find an equation of the tangent line to the curve of intersection of these surfaces at the point $(1, 1, 1)$.

Solution:

a. The surface is a level set of the function $F(x, y, z) = 2x^2 + 3y^2 + 4z^3$ and so the normal to the tangent plane is $\nabla F(1, 1, 1)$. Since

$$\nabla F = (4x, 6y, 12z^2)$$

the normal is $(4, 6, 12)$ equivalently $(2, 3, 6)$. The equation of the tangent plane is $2(x - 1) + 3(y - 1) + 6(z - 1) = 0$.

b. The surface is the graph of the function $f(x, y) = 2x^2 - y^3$. So, the normal to the tangent plane is $(f_x, f_y, -1) = (4x, -3y^2, -1)$ which evaluates to $(4, -3, -1)$ at the given point. The equation of the tangent plane is $4(x - 1) - 3(y - 1) - (z - 1) = 0$.

Note: this surface can also be considered a level set of a function of 3 variables.

c. A vector parallel to the tangent line of the curve of intersection is $(2, 3, 6) \times (4, -3, -1) = (15, 26, -18)$. Thus an equation of the tangent line is:

$$\frac{x - 1}{15} = \frac{y - 1}{26} = \frac{1 - z}{18}$$

2.

(a) Let A be an $m \times n$ matrix. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$F(\mathbf{x}) = A\mathbf{x}^T$$

Find the Jacobian, DF, of F, in terms of A.

(b) Let B be a *symmetric* $n \times n$ matrix. Let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$g(\mathbf{x}) = \mathbf{x}B\mathbf{x}^T$$

Find the Hessian of g in terms of B.

Solution:

a. Write

$$F(x_1, \ldots, x_n) = [F_1(x_1, \ldots, x_n), \ldots, F_m(x_1, \ldots, x_n)]$$

Thus, the ij entry of the Jacobian DF is

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{k=1}^{n} A_{ik} x_k = A_{ij}.$$

Thus, $DF = A$.

b.

$$g(\mathbf{x}) = \mathbf{x}B\mathbf{x}^T = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} x_i x_j$$

Thus,

$$\frac{\partial g}{\partial x_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \frac{\partial (x_i x_j)}{\partial x_k} = \sum_{j=1,\ j \neq k}^{n} B_{kj} x_j + \sum_{i=1,\ i \neq k}^{n} B_{ik} x_i + 2B_{kk} x_k.$$

So

$$\frac{\partial^2 g}{\partial x_\ell \partial x_k} = \left\{ \begin{array}{cc} B_{k\ell} + B_{\ell k} & \ell \neq k \\ 2B_{kk} & \ell = k \end{array} \right\} = 2B.$$

by symmetry.

3.

Evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\; dydx$$

Solution: The domain is the quarter disk in the first quadrant centered at the origin of radius 2. Switch to polar coordinates:

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{2} r^2 \; drd\theta$$

$$= 4\pi/3.$$

4.

Re-iterate the integral

$$\int_{x=1}^{2}\int_{y=x}^{2}\int_{z=\log x}^{\log y} f(x,y,z)\; dzdydx$$

in the following orders by filling in the upper and lower limits of the integrals.

(a)

$$\int\int\int f(x,y,z)\; dxdzdy$$

(b)

$$\int\int\int f(x,y,z)\; dydxdz$$

NO justification required.

Solution:

$1 \le x \le 2$

$x \le y \le 2$

$\log x \le z \le \log y$

$1 \le y \le 2$

$0 \le z \le \log y$

$1 \le x \le e^z$

$0 \le z \le \log 2$

$1 \le x \le e^z$

$e^z \le y \le 2$

5.

Evaluate $\int\int\int_D \; x^2 + y^2 + z^2 \; dV$ where D is the solid lying inside the sphere of radius 1 centered at $(0, 0, 1)$ and inside (i.e above) the cone $x^2 + y^2 = 3z^2$.

Solution:

The domain is the intersection of the solids

$$x^2 + y^2 + (z-1)^2 \leq 1$$

equivalently,

$$x^2 + y^2 + z^2 \leq 2z$$

and

$$x^2 + y^2 \leq 3z^2$$

Substituting spherical coordinate representations for x, y, z we get

$$\rho^2 \leq 2\rho\cos\phi, \quad \rho^2\sin^2\phi \leq 3\rho^2\cos^2\phi$$

This is equivalent to

$$\rho \leq 2\cos\phi, \quad 0 \leq \phi \leq \pi/3.$$

Thus the integral becomes

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=0}^{2\cos\phi} \rho^4 \sin\phi \; d\rho d\phi d\theta$$

$$(2\pi/5)\int_{\phi=0}^{\pi/3} \rho^5 \sin\phi\Big]_{\rho=0}^{2\cos\phi} d\phi$$

$$= (2\pi/5)\int_{\phi=0}^{\pi/3} (2\cos\phi)^5 \sin\phi d\phi =$$

$$= (64\pi/5)\int_{u=1/2}^{1} u^5 du = (64\pi/30)(1-1/2^6)$$

6.

Give $\epsilon - \delta$ proofs to justify the following statements.

(a) $f(x, y) = \sqrt{x^2 + y^4}$ is continuous at $(0, 0)$.

(b) $g(x, y) = x^2 y$ is differentiable at $(1, 1)$.

For part b, you do NOT need to justify computation of partial derivatives with $\epsilon - \delta$ proofs.

Solution:

a. If $\sqrt{x^2 + y^2} < \delta$, then $|x| < \delta$ and $|y| < \delta$. Thus

$$\sqrt{x^2 + y^4} \leq \sqrt{\delta^2 + \delta^4} = \delta\sqrt{1 + \delta^2}$$

which is $< \sqrt{2}\delta$ if $\delta < 1$.
Choose $\delta = \min(1, \epsilon/\sqrt{2})$.

Alternative. If $\sqrt{x^2 + y^2} < \delta < 1$, then $|y| < 1$ and so $y^4 \leq y^2$. So, $\sqrt{x^2 + y^4} \leq \sqrt{x^2 + y^2} < \delta$.
Choose $\delta = \min(1, \epsilon)$.

b. $g_x = 2xy, g_y = x^2$. So, $g_x(1, 1) = 2, g_y(1, 1) = 1$.
So, we must verify that

$$\lim_{(x,y)\to(1,1)} \frac{x^2 y - 1 - 2(x - 1) - (y - 1)}{\sqrt{(x - 1)^2 + (y - 1)^2}} = 0.$$

The numerator can be written as

$$x^2 y - y - 2x + 2 = y(x - 1)(x + 1) - 2(x - 1) = (x - 1)(y(x + 1) - 2)$$

Thus,

$$\left|\frac{x^2 y - 1 - 2(x - 1) - (y - 1)}{\sqrt{(x - 1)^2 + (y - 1)^2}}\right| = \frac{|x - 1|}{\sqrt{(x - 1)^2 + (y - 1)^2}}|y(x + 1) - 2|$$

The first factor is bounded above by 1 and the second factor may be re-written

$$|(y-1)(x+1)+(x-1)| \leq |y-1||x+1|+|x-1|$$

So, if $\sqrt{(x-1)^2+(y-1)^2}<\delta$ and $\delta<1$, the second factor is bounded by 4δ. So, choose $\delta=\min(1,\epsilon/4)$.

7. (a) Show that the intersection of any two open sets is open.

   (b) Show that the intersection of any two closed sets is closed.

   (c) Show that the boundary of any set is closed.

   Solution:

   a. Let U and V be open. Let $x \in U \cap V$. Since U and V are open, there exist $r > 0$ and $s > 0$ such that $B_r(x) \subseteq U$ and $B_s(x) \subseteq V$. Let $t = \min(r, s)$. Then $B_t(x) \subseteq B_r(x) \cap B_s(x) \subseteq U \cap V$.

   b. Since $(A \cap B)^c = A^c \cup B^c$, it suffices to show that the union of any two open sets is open. For this, let U and V be open and $x \in U \cup V$. Without loss of generality, we may assume $x \in U$. Then there exists $r > 0$ such that $B_r(x) \subseteq U \subseteq U \cup V$.

   c. Let D be any set. It suffices to show that $(\partial D)^c$ is open. For this, let $x \in (\partial D)^c$. Then there exists $r > 0$ such that either $B_r(x) \cap D = \emptyset$ or $B_r(x) \cap D^c = \emptyset$.

   In the former case $B_r(x) \subseteq D^c$. We show that $B_r(x) \subseteq (\partial D)^c$. For any $y \in B_r(x)$, let $s = r - d(x, y) > 0$. Then by the triangle inequality, $B_s(y) \subseteq B_r(x) \subseteq D^c$. Thus, $y \in (\partial D)^c$.

   In the latter case $B_r(x) \subseteq D$. We show that $B_r(x) \subseteq (\partial D)^c$. For any $y \in B_r(x)$, let $s = r - d(x, y) > 0$. Then by the triangle inequality, $B_s(y) \subseteq B_r(x) \subseteq D$. Thus, $y \in (\partial D)^c$.

8.

Find the absolute maxima and absolute minima (and their values) of the function

$$f(x,y,z) = x\log x + y\log y + z\log z$$

subject to the constraints $x+y+z=1$, $x \geq 0, y \geq 0, z \geq 0$.

You may assume $0\log 0 = 0$.

Solution:

Set up as a Lagrange multipliers problem:

The constraint conditions define the surface which is the union of the interior and boundary of the triangle with vertices (1,0,0), (0,1,0) and (0,0,1).

Let $g(x,y,z) = x+y+z$. The Lagrangian is

$$L(x,y,z) = x\log x + y\log y + z\log z + \lambda(x+y+z-1)$$

The critical points of the Lagrangian are given by:

$$\begin{aligned} A: &\quad \log x + 1 + \lambda = 0 \\ B: &\quad \log y + 1 + \lambda = 0 \\ C: &\quad \log z + 1 + \lambda = 0 \\ D: &\quad x+y+z = 1 \end{aligned}$$

From A, B, and C, we find that $x = y = z$. From D, we find that $x = y = z = 1/3$. And $f(1/3, 1/3, 1/3) = -\log 3$

There are no singular points in the interior of the triangle and $\nabla g = (1,1,1)$ is of course never 0. So, it remains to examine the boundary points ("endpoints"). By symmetry we need only consider one side of the triangle, say the line from (1,0,0) to (0,1,0). On this line $f(x,y,z) = x\log x + y\log y$ (recall that for $z=0$, $z\log z = 0$).

This amounts to extremizing the function $g(x) = x \log x + (1-x)\log(1-x)$ on $0 \leq x \leq 1$.

Now, $g'(x) = \log x + 1 - \log(1-x) - 1$ which is 0 exactly when $x = 1-x$, i.e., $x = 1/2$. And $g(1/2) = -\log 2$. So, a candidate for an extremum is $(1/2, 1/2, 0)$ and $f(1/2, 1/2, 0) = -\log 2$

Finally, we must consider the vertices $(1, 0, 0)$ and $(0, 1, 0)$ and $(0.0.1)$, and $f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1)0$.

Since $-\log 3 < -\log 2 < 0$, the absolute minimum is $(1/3, 1/3, 1/3)$ with absolute maximum value $-\log 3$ and the absolute maxima are the vertices with f-values $= 0$. This assumes that absolute max and min exist. But this is true since f is a continuous function on a closed bounded set (for continuity, one uses L'Hopital's rule).