Department of Mathematics
University of British Columbia
MATH 226 Final Exam
December 11, 2012, 12:00PM - 2:30PM
SOLUTIONS

Family Name: ______________________________ Initials: ______

I.D. Number: ________________ Signature: ______________________________

CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.

JUSTIFY ALL OF YOUR ANSWERS (except as otherwise specified).

THERE ARE 8 PROBLEMS ON THIS EXAM.

Question	**Mark**	**Out of**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		80

1.

Let $f(x,y) = x^2 - 2xy + 3y^2$. Let $(x_0, y_0) = (1,2)$,

(a) At the point (x_0, y_0), find the direction in which f increases most rapidly.

(b) At the point (x_0, y_0), find the set directions in which the rate of increase of f is at least 1/2 of the maximum rate of increase. Give your answer in terms of the range of angles of deviation from the direction of maximum increase.

(c) Find an equation of the tangent line to the level curve of f passing through (x_0, y_0).

Solution:

a. $\nabla f = (2x - 2y, -2x + 6y)$. At $(1,2)$, $\nabla f = (-2, 10)$. Direction of maximal increase is $(-1,5)/\sqrt{26}$.

b,

$$|\nabla f| \cos(\theta) = (\nabla f) \cdot \mathbf{u} = D_{\mathbf{u}} f \geq (1/2)|\nabla f|$$

Equivalently, $\cos(\theta) \geq 1/2$ and so the range of angles is $\pm$ 60 degrees. (Note that this has nothing to do with the specific function f).

c. Level set is orthogonal to the gradient. So, a tangent vector to the line is $(5,1)$. Thus, $(x,y) = (1,2) + t(5,10)$ is a (parametric) equation of the line.

2.

For each of the following subsets of $\mathbb{R}^3$, determine if the set is:

- open
- closed
- bounded

No justification required.

Enter Y for Yes and N for No in the table.

	a	b	c	d	e	f	g	h	i	j
open	Y	N	N	N	Y	N	N	N	Y	N
closed	N	Y	Y	N	Y	Y	Y	Y	N	Y
bounded	N	Y	N	Y	N	Y	N	N	N	N

(a) $\{(x,y,z) \in \mathbb{R}^3 :\ 0 < x^2 + y^2 + z^2\}$

(b) $\{(x,y,z) \in \mathbb{R}^3 :\ 3x^2 + 4y^2 + 5z^2 = 1\}$

(c) $\{(x,y,z) \in \mathbb{R}^3 :\ 3x^2 + 4y^2 - 5z^2 = 1\}$

(d) $\{(x,y,z) \in \mathbb{R}^3 :\ 2 < x^2 + y^2 + z^2 \leq 3\}$

(e) $\mathbb{R}^3$

(f) $\{(1,0,0),(0,1,0),(0,0,1),(1,-4,2)\}$

(g) $\{(x,y,z) \in \mathbb{R}^3 :\ y = 0\}$

(h) The x-axis

(i) The complement of the x-axis

(j) The complement of $\{(x,y,z) : x > 0\}$

3.

Compute

$$\int\int_D xy\ dxdy$$

where D is the domain

$$x^2+y^2+x \leq \sqrt{x^2+y^2}, \quad y \geq 0$$

Solution: Change to polar coordinates:

$$r^2 + r\cos\theta \leq r, \quad 0 \leq \theta \leq \pi$$

equivalently,

$$r \leq 1-\cos\theta, \quad 0 \leq \theta \leq \pi$$

(this is the upper half of a cardioid).

So, the integral becomes

$$\int_{\theta=0}^{\pi}\int_{r=0}^{1-\cos\theta} r^2(\cos\theta\sin\theta)rdrd\theta =$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{1-\cos\theta} r^3(\cos\theta\sin\theta)drd\theta$$

$$= (1/4)\int_{\theta=0}^{\pi}(1-\cos\theta)^4(\cos\theta\sin\theta)d\theta$$

Set $u = (1-\cos\theta)$, and we get

$$= (1/4)\int_{u=0}^{2} u^4 - u^5du = (1/4)(u^5/5 - u^6/6)]_0^2 = (1/4)(2^5/5 - 2^6/6)$$

4.

Compute

$$\int\int_D x^2y\log y + 2x^2y\ dxdy$$

where D is the domain

$$e^x \le y \le e^{2x},\ \ 1 \le x^2y \le 2.$$

Note: log is the natural logarithm.

Solution:

This region is in the first quadrant, bounded by 4 curves:

$$(\log y)/x = 1, (\log y)/x = 2, x^2y = 1, x^2y = 2.$$

Change Of Variables (this is the same principle as Example 8, p. 832):

$$u = (\log y)/x,\ \ 1 \le u \le 2,$$

$$v = x^2y,\ \ 1 \le v \le 2.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{bmatrix} -(\log y)/x^2 & 1/(xy) \\ 2xy & x^2 \end{bmatrix} \right| = -(2 + \log y)$$

Claim: $2 + \log y > 0$.

Proof: $x^2 \ge 1/y \ge e^{-2x}$ And so $xe^x \ge 1$ and so $x \ge e^{-x} > 0$. Thus, $y \ge e^x > 1$ And so $\log y > 0$.

Thus,

$$\int\int_D x^2y\log y + 2x^2y\ dxdy$$

$$= \int_1^2\int_1^2 (x^2y\log y + 2x^2y)(1/(2+\log y))\ dudv$$

$$= \int_1^2\int_1^2 v\ dudv = 3/2.$$

Note: there was discussion in the review session on whether this domain is entirely contained in the first quadrant. To see this:

Since $1 \leq x^2y$, we have $y > 0$; since $e^x \leq y \leq e^{2x}$, we have $1 \leq e^x$ and so $x \geq 0$ and from $1 \leq x^2y$, we then have $x > 0$.

5.

Re-write the integral

$$\int_{z=0}^{1} \int_{y=0}^{\sqrt{z}} \int_{x=0}^{\sqrt{z-y^2}} f(x,y,z)\ dxdydz$$

as

(a)

$$\int_{x=\square}^{\square} \int_{z=\square}^{\square} \int_{y=\square}^{\square} f(x,y,z)\ dydzdx$$

(b)

$$\int_{y=\square}^{\square} \int_{x=\square}^{\square} \int_{z=\square}^{\square} f(x,y,z)\ dzdxdy$$

NO justification required.

Solution:

a.

$0 \leq x \leq 1$

$x^2 \leq z \leq 1$

$0 \leq y \leq \sqrt{z - x^2}$

Note: the lower bound in the second inequality comes from: $x \leq \sqrt{z - y^2}$ (in the original integration) which implies $x^2 \leq x^2 + y^2 \leq z$. The upper bound in the last inequality comes from $x \leq \sqrt{z - y^2}$ which implies $y^2 \leq z - x^2$ which implies $y \leq \sqrt{z - x^2}$, which is a tighter constraint than $y \leq \sqrt{z}$ (in the original integration).

b.

$0 \leq y \leq 1$

$0 \leq x \leq \sqrt{1-y^2}$

$x^2 + y^2 \leq z \leq 1$

Note: the upper bound in the second inequality comes from $x \leq \sqrt{z-y^2}$ an $0 \leq z \leq 1$ (in the original integration), and the lower bound in the last inequality is tighter than the lower bound $y^2 \leq z$ (which comes from $y \leq \sqrt{z}$ in the original integration).

6.

In this problem, give careful arguments, but you do NOT need to give $\epsilon - \delta$ proofs.

(a) Let

$$f(x,y) = \begin{Bmatrix} \frac{x\sin(x^2+y^2)+y^3-xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{Bmatrix}$$

i. Compute the partial derivatives of f at (0,0).

ii. At (0,0), is f continuous? differentiable? continuously differentiable ?

(b) For each *real* number $r \geq 0$, consider the function

$$g(x,y) = \begin{Bmatrix} (x^2+y^2)^r \sin(\frac{1}{x^2+y^2}) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{Bmatrix}$$

Find all values of $r \geq 0$ such that at $(0,0)$, g is

i. continuous

ii. differentiable

iii. continuously differentiable

Solution:

a. i.

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 1.$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{y^3}{y^3} = 1.$$

ii. Continuity: Yes:

$$\left|\frac{x\sin(x^2+y^2)+y^3-xy^2}{x^2+y^2}\right| \leq |x|\left|\frac{\sin(x^2+y^2)}{x^2+y^2}\right| + |y|\frac{y^2}{x^2+y^2} + |x|\frac{y^2}{x^2+y^2}$$

The first term tends to $0 \cdot 1 = 0$; the second and third terms tend to 0 since $\frac{y^2}{x^2+y^2} \leq 1$.

Differentiability: No. Check to see if the following limit is 0:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (f_x(0,0))x - (f_y(0,0))y}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2) + y^3 - xy^2 - x(x^2+y^2) - y(x^2+y^2)}{(x^2+y^2)^{3/2}}$$

Set $x = y$ and let $x \to 0^+$ and we get:

$$\lim_{x\to 0^+} \frac{x\sin(2x^2) - 4x^3}{(\sqrt{2}x)(2x^2)} = (1/\sqrt{2}) - \sqrt{2} \neq 0.$$

Continuous Differentiability: No, since continuous differentiability implies differentiability.

b. A homework problem: HW5 #3.

7.

Prove that the following functions are continuous everywhere by using the $\epsilon - \delta$ definition. Do not use any other properties of limits or continuous functions.

(a) $f(x,y) = \frac{1}{x^2+y^2+1}$

(b) $f(x,y,z) = xyz$

a. Consider continuity at (a,b).

$$\left|\frac{1}{x^2+y^2+1} - \frac{1}{a^2+b^2+1}\right| = \frac{|a^2-x^2+b^2-y^2|}{(x^2+y^2+1)(a^2+b^2+1)}$$

$$\leq |a-x||a+x| + |b-x||b+x|$$

If $\sqrt{(a-x)^2+(b-y)^2} < \delta$, then $|x-a| < \delta$ and $|y-b| < \delta$. Also if $\delta < 1$, then

$$|x+a| = |(x-a)+2a| \leq |x-a| + 2|a| < 2|a| + 1$$

So $|x+a| < 2|a|+1$ and similarly $|y+b| < 2|b|+1$

So

$$|a-x||a+x| + |b-x||b+x| < 2\delta(|a|+|b|+1)$$

letting $\delta = \min(1, \frac{\epsilon}{2(|a|+|b|+1)})$ we obtain

$$\left|\frac{1}{x^2+y^2+1} - \frac{1}{a^2+b^2+1}\right| < \epsilon$$

This proves continuity.

b. An homework problem: HW4 #2a.

8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Assume that for all (x, y, z), $\nabla f(x, y, z)$ is parallel to the vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that f is a function of $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, i.e., there is a function F such that for all (x, y, z), $\quad f(x, y, z) = F(\rho(x, y, z))$.

Solution:

Let $T(\rho, \phi, \theta) = (x, y, z)$ be the spherical coordinates transformation. Let $g = f \circ T$. It suffices to show that $\frac{\partial g}{\partial \phi} = 0$ and $\frac{\partial g}{\partial \theta} = 0$; for then g is constant as a function of both ϕ and θ and so g is a function of ρ and so f is a function of $\sqrt{x^2 + y^2 + z^2}$.

For this, apply the chain rule:

$$Dg = Df DT$$

Now,

$$Df = \nabla f = (\lambda(x, y, z))(x, y, z)$$

for some scalar function $\lambda(x, y, z)$. So,

$$Dg = Df DT = (\lambda(x, y, z))(x, y, z)DT$$

$$= (\lambda(x, y, z))(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))DT$$

This is a vector with three components. Using the formula for the Jacobian DT at top of p. 845 a straightforward calculation shows that this vector 0 in the last two components, which are precisely the partial derivatives of g with respect to ϕ and θ.

As pointed out in the review session, another approach is to show that the directional derivative of f in any direction tangent to a sphere centered at the origin is zero. There are more details to complete the proof, but this approach has the advantage of being much more geometric.