3. Consider the function $f(x, y) = \ln\left(a + \frac{y}{x^2}\right)$, where a is a parameter.

[4 marks] (a) Determine the domain and range of the function f (consider the cases $a = 0$, $a > 0$, and $a < 0$ separately).

[4 marks] (b) Suppose $a = -1$. Is the domain of f a closed set, an open set, or neither? (Include brief proof using the definition of an open/closed set).

[4 marks] (c) What do the level curves $f(x, y)$ look like in each of the cases? Describe and sketch them.

(a) For ln to be defined, we need $a + \frac{y}{x^2} > 0$

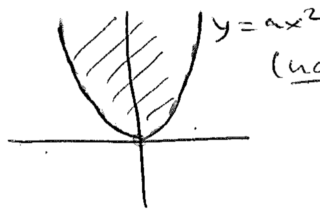
$\Leftrightarrow \frac{y}{x^2} > -a$

$\Leftrightarrow \begin{cases} x \neq 0 \\ y > -ax^2 \end{cases}$

The domain is defined by:

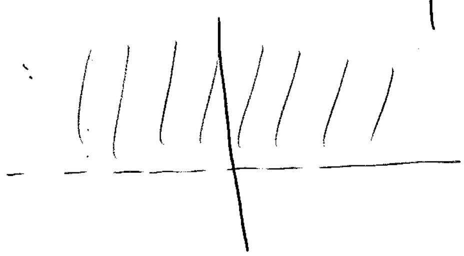
$$\boxed{D = \{(x, y) \mid x \neq 0 \text{ and } y > -ax^2\}}$$

Sketch: $a > 0$



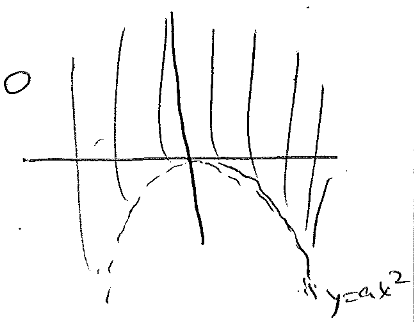
(not including the boundary)

$a = 0$:



$(y > 0)$

$a < 0$



Range: $s \in \text{Range}(f)$

$\Leftrightarrow \exists (x, y)$ s.t. $\ln\left(a + \frac{y}{x^2}\right) = s$

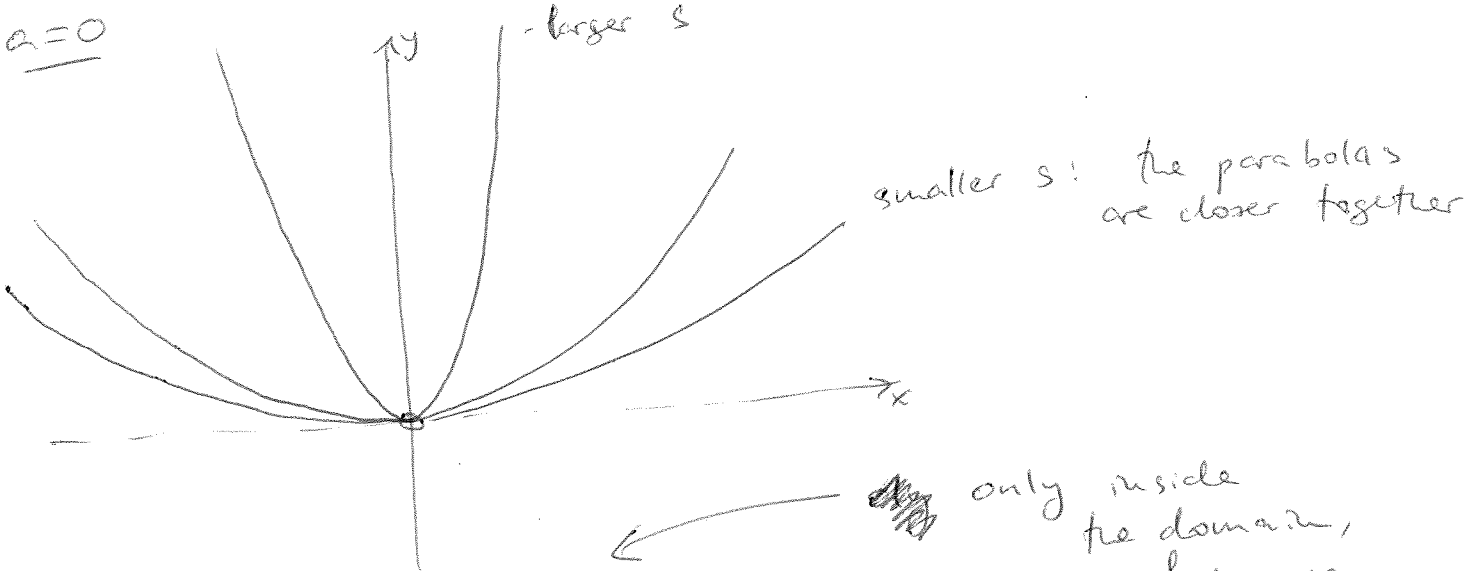
$\Leftrightarrow a + \frac{y}{x^2} = e^s$

$\Leftrightarrow \frac{y}{x^2} = e^s - a$

$\Leftrightarrow y = (e^s - a)x^2$ – the set defined by this equation is non-empty for all $s \in \mathbb{R}$.

Answer Range $(f) = \mathbb{R}$.

(c) As proved in (a), the level curves have equations $y = (e^s - a)x^2$ so they are all parabolas.



only inside the domain, of course