3. Consider the function \( f(x, y) = \ln \left( a + \frac{y}{x^2} \right) \), where \( a \) is a parameter.

4 marks

(a) Determine the domain and range of the function \( f \) (consider the cases \( a = 0 \), \( a > 0 \), and \( a < 0 \) separately).

4 marks

(b) Suppose \( a = -1 \). Is the domain of \( f \) a closed set, an open set, or neither? (Include brief proof using the definition of an open/closed set).

4 marks

(c) What do the level curves \( f(x, y) \) look like in each of the cases? Describe and sketch them.

(a) For \( \ln \) to be defined, we need \( a + \frac{y}{x^2} > 0 \).

The domain is defined by:

\[
D = \left\{ (x, y) \mid x \neq 0 \text{ and } y > -ax^2 \right\}
\]

Sketch: \( a > 0 \)

\[
y = ax^2
\]

(not including the boundary)

\( a = 0 \): \[
(\text{y} > 0)
\]

\( a < 0 \): \[
(\text{y} > 0)
\]

Range: \( \subseteq \text{Range}(f) \)

\[
(\Rightarrow) \exists (x, y) \text{ s.t. } \ln \left( a + \frac{y}{x^2} \right) = s
\]

\[
(\Rightarrow) a + \frac{y}{x^2} = e^s
\]

\[
(\Rightarrow) \frac{y}{x^2} = e^s - a
\]

\[
(\Rightarrow) y = (e^s - a)x^2
\]

The set defined by this equation is non-empty for all \( s \in \mathbb{R} \).

Answer: \( \text{Range } (f) = \mathbb{R} \).
(c) As proved in (a), the level curves have equations \( y = (e^a - a)x^2 \) so they are all parabolas.

\[ a = 0 \]

- Larger \( s \)

- Smaller \( s \): the parabolas are closer together

\[ a < 0 \]

\[ y = -ax^2 \]

\[ a > 0 \]

- More shallow and closer together for smaller \( s \)

- \( e^a > a \)

- \( e^a < a \)