2. *(In this problem, the parts are related. If you could not do an earlier part, you can give some notation to the answer for it, and proceed to the later parts using that notation, to get partial credit).*
We make the convention that he x-axis points East, the y-axis points North, and the z-axis points up. The unit on each axis is 1km. Suppose a small airplane is moving along a straight line L_1. At $t = 0$, it is at the point $(0, 0, 3)$ (that is, its altitude is 3km). The projection of its trajectory onto the xy-plane points directly North-East, its altitude is increasing at 100m/min, and its ground speed (i.e. the speed at which its projection to the ground is moving) is 10km/min.

3 marks (a) Find the velocity and speed of the airplane (velocity is a 3d-vector, and speed is the magnitude of the velocity).

3 marks (b) Find the parametric equations for the line L_1 (the trajectory of the airplane).

2 marks (c) Suppose another airplane is flying along the line L_2 with the parametric equation $\mathbf{r}(t) = \langle 0, 10t, 2\sqrt{2}\rangle$. Find the symmetric equations for L_2.

5 marks (d) Does there exist a plane that contains the trajectories of the both airplanes? If yes, give its equation.

Solution

(a) Let $\bar{v}$ be the velocity of the plane.
Since the projection of the plane's trajectory onto the xy-plane points NE, we know that the projection of $\bar{v}$ onto the xy-plane is parallel to the vector $\langle 1, 1\rangle$.
Its magnitude is 10 (km/min).
Then its components are $\langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}\rangle$.
Now, the projection of $\bar{v}$ onto the z-axis is the vector $\langle 0, 0, 0.1\rangle$ (because the altitude is increasing at 0.1 km/min)

Thus, $\boxed{\bar{v} = \langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 0.1\rangle}$ - velocity.

speed $= |\bar{v}| = \sqrt{10^2 + (0.1)^2} = \boxed{\sqrt{100.01} \text{ km/min}}$

(b) Use $\bar{v}$ from part (a) as the direction vector.
Answer: $\bar{r}(t) = \langle \frac{10}{\sqrt{2}}t, \frac{10}{\sqrt{2}}t, 3 + 0.1t\rangle$

$$\begin{cases} x(t) = \frac{10}{\sqrt{2}}t \\ y(t) = \frac{10}{\sqrt{2}}t \\ z(t) = 3 + 0.1t \end{cases}$$

(c) From the vector-parametric equation, we get

$$\begin{cases} x = 0 \\ z = 2\sqrt{2} \end{cases}$$

– this is the symmetric equation (in our case, the direction vector has two zero components).

(d) We need to check if the vector connecting two points on these trajectories lies in the plane spanned by their direction vectors.

We have: direction vectors are

$$\bar{v}_1 = \left\langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}, 0.1 \right\rangle$$

$$\bar{v}_2 = \langle 0, 10, 0 \rangle$$

We can make a vector connecting their two initial points (at $t=0$):

$P_1 = (0,0,3)$ $\quad$ $P_2 = (0,0,2\sqrt{2})$

The vector $\overline{P_1P_2}$ is parallel to the z-axis.

We need to check if it is perpendicular to ~~[illegible]~~ $\bar{v}_1 \times \bar{v}_2$

(this is the same as checking whether the plane containing $\bar{v}_1$ and $\bar{v}_2$ is horizontal).

We can compute the cross product of $\bar{v}_1$ and $\bar{v}_2$ and see that it is <u>not</u> perpendicular to the z-axis; or we can just note that the line L_1 does <u>not</u> lie in a horizontal plane.

In any case, the answer is: L_1 and L_2 are skew, and there is no plane containing both.