Last time: Basis Extension Theorem

Recall $\{v_1, \dots, v_k\}$ - lin. indep. vectors in V

$\{w_1, \dots, w_s\}$ - some other vectors in V s.t.

$L(v_1, \dots, v_k, \underbrace{w_1, \dots, w_s}_{\text{red}}) = V$, then you can choose

some of the w's, s.t. together with the v's

they form a basis of V.

Today: consequences from this.

① Basis exchange lemma

Suppose $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$

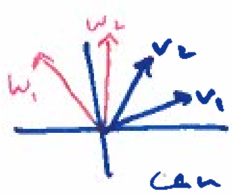
are both bases of a vector space V. [should be the same as n but we have not proved it yet

Then v_1 can be replaced with

one of the w's, and

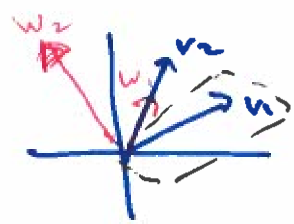
$\{w_i, v_2, \dots, v_n\}$ will still be a basis.

Picture:



can make a basis out of one blue vector and one red vector.

less obvious case:



Proof: We are trying to exchange v_1 for one of the w's

Consider $L(v_2, \dots, v_n) \subset V$

Claim: there exists i, s.t. $w_i \notin L(v_2, \dots, v_n)$

at least one of the w's is not in this subspace.

Pf of the claim: If all w_i lie in $L(v_2, \dots, v_n)$

then $L(w_1, \dots, w_m) \subseteq L(v_2, \dots, v_n)$

But $\{w_1, \dots, w_m\}$ form a basis of the whole space V, so this is impossible (unless $L(v_2, \dots v_n) = V$, but this cannot happen because $\{v_1, \dots, v_n\}$ was a basis, so linearly indep.; and if $L(v_2, \dots, v_n) = V$, then $\{v_1, v_2 \dots v_n\}$ would have been linearly dependent).

So let i be such an index that $w_i \notin L(v_2, \dots, v_n)$.

Then <u>claim</u>: $\{w_i, v_2, \dots v_n\}$ is a basis of V.

We know: $\exists\ \lambda_1, \dots, \lambda_n \in F$ s.t.

$$w_i = \lambda_1 v_1 + \dots + \lambda_n v_n$$

and $\lambda_1 \neq 0$ because otherwise w_i would be in $L(v_2, \dots, v_n)$

Then $$v_1 = \frac{1}{\lambda_1}(w_i - \lambda_2 v_2 - \dots - \lambda_n v_n)$$

Then $V = L(v_1, \dots, v_n) \subseteq L(w_i, v_2, v_3, \dots, v_n)$

Then $\{w_i, v_2, \dots, v_n\}$ is a spanning set for V, and it is lin. indep. b/c $w_i \notin L(v_2, \dots, v_n)$ and $v_2 \dots v_n$ are independent.

So it is a new basis.

Proposition: 1) If V has a basis of n elements, then every basis of V has n elements.

$\boxed{\dim(V) = n}$

2) Any collection of vectors of V that has more than n elements is linearly dependent.

Pf: 2) Suppose $\{v_1, \dots, v_n\}$ is a basis

Suppose ~~set~~ that $\{w_1, \dots w_{n+1}\}$ is a lin. ~~indep~~ indep. collection of vectors

Then $\{w_1, \dots w_{n+1}, \underbrace{v_{s_1}, v_{s_2}, \dots v_{s_k}}\}$ is a basis for some $s_1, \dots, s_k$ by the basis extension Thm

put some of the v's together with w's, if needed.

Now I can exchange w_1 for one of the v's by the exchange lemma. Will get a new basis.
Keep exchanging untill all the w's are gone.
Will get ~~a~~ a ~~set~~ list of vectors where some of the v's are listed more than once!
This cannot be a linearly independent collection!
We got a contradiction — there could not have been $n+1$ lin. indep. w's.

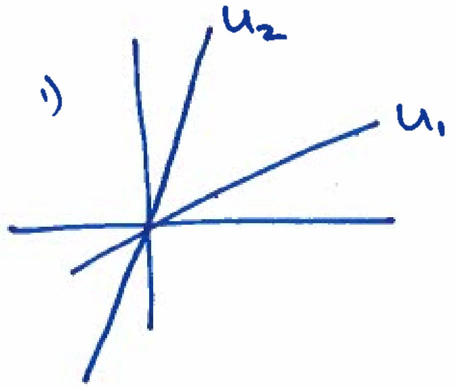
$(2) \Rightarrow (1)$.

Def: Let U_1, U_2 - linear subspaces of V.

$$U_1 + U_2 = \{ x+y \mid x \in U_1, y \in U_2 \}.$$

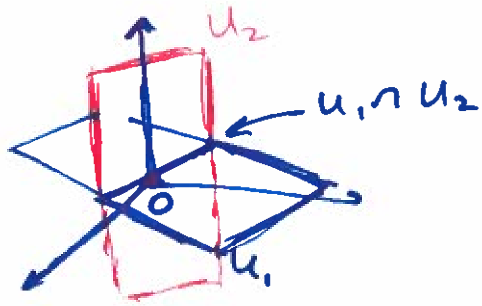
Exer: check that $U_1 + U_2$ is a linear subspace of V.

Examples: 1)



$U_1 + U_2 = \mathbb{R}^2$

2) in $\mathbb{R}^3$:



$U_1 + U_2 = \mathbb{R}^3$.

Prop: Let U_1, U_2 be linear subspaces of V. assume U_1, U_2 are finite-dimensional.

Then

$$\boxed{\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2)}$$

check: $U_1 \cap U_2$ is a lin. sub.

Check: in our example in $\mathbb{R}^3$:

$\dim(U_1) = \dim(U_2) = 2$

$\dim(U_1 \cap U_2) = 1$.

$\dim(U_1 + U_2) = 3$.

Question: in $\mathbb{R}^4$, do two planes containing 0 have to have a common line?

Yes: $\mathbb{R}^4 = L(e_1, \dots, e_4)$. Take $U_1 = L(e_1, e_2)$

$U_2 = L(e_3, e_4)$

Def: If $U_1 \cap U_2 = \{0\}$ and $U_1 + U_2 = V$ then we say that V is a 'direct sum' of U_1 and U_2, write $V = U_1 \oplus U_2$

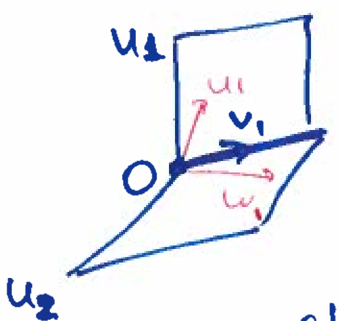
\oplus

In our $\mathbb{R}^4$ example, check:

$$\mathbb{R}^4 = L(e_1, e_2) \oplus L(e_3, e_4)$$

Pf of the dimension formula

Consider $U_1 \cap U_2$; let $v_1, \dots, v_s$ be its basis.



Use basis extension to extend it to $\{v_1, \dots, v_s, u_1, \dots u_\ell\}$ – basis of U_1

$\{v_1 \dots v_s, w_1 \dots w_k\}$ – basis of U_2

<u>Claim</u>: $\{v_1, \dots v_s, u_1, \dots u_\ell, w_1 \dots w_k\}$ is a basis of $U_1 + U_2$.

exer

The formula follows:

$$\dim(U_1 + U_2) = s + \ell + k$$

$$\dim(U_1) + \dim(U_2) = (s + \ell) + (s + k).$$
