<u>Recall</u> last time:

- linear combinations
- <u>basis</u> of V: a set of vectors $v_1, \ldots, v_n$ in V both <u>spanning</u>: $L(v_1, \ldots v_n) = V$ and <u>linearly independent</u>.

- <u>Theorem</u>: Every basis has the same number of elements (when a finite basis exists).

<u>Today</u>: . examples
. prove this theorem (will need some lemmas).

<u>Examples</u>: ① Let V be a vector space of dimension n ($\dim V = n$) over $\mathbb{F}_p$ ← field of p elements.

How many elements does V have?

<u>Warm-up</u>: $\dim V = 1$;
Then $\#V = p$
↗
"cardinality"

$$\boxed{\begin{array}{l} \text{If } \dim V = 0 \text{ then } \#V = 1 \\ \Downarrow \quad\quad \text{over any} \\ V = \{0\} \quad \text{field.} \end{array}}$$

Over any field, a 1-dim. vector space is $\mathbb{F} \cdot v_1$
↑
basis element of V

it looks like $\mathbb{F}$ itself
(think of $\mathbb{F}$ as a vector space over itself, dimension is 1
basis: any $x \in \mathbb{F}$, $x \neq 0$ gives a basis:

$$L(x) = \{ y \cdot x \mid y \in \mathbb{F} \} = \mathbb{F} \quad)$$

$\dim V = 2$, V over $\mathbb{F}_p$.

Standard basis: $V = \mathbb{F}_p^n$ ← elements of $\mathbb{F}_p^n$ as a set
(canonical)

basis $e_1 = (1, 0, 0 \dots 0)$
$e_2 = (0, 1, 0 \dots, 0)$
— —
$e_n = (0, 0 \dots, 1)$

Why do they form a basis?

- check: 1) lin. indep.
  2) spanning. ← exer.

1) Suppose $\lambda_1 e_1 + \dots + \lambda_n e_n = 0$. want to show: $\lambda_1 = \dots = \lambda_n = 0$.

$$\lambda_1(1, 0 \dots 0) + \dots + \lambda_n(0, \dots, 1)$$
$$= (\lambda_1, \lambda_2, \dots, \lambda_n) \leftarrow \text{if this is } \bar{0}, \text{ then all } \lambda_i = 0.$$

Main point: ~~uses~~ $\mathbb{F}^n$ comes with a basis $\{e_1, \dots, e_n\}$

point for the future: any n-dim. vector space over F is "isomorphic" to F^n. ("the same as")

— —

So: $\# V$ if $\dim V = n$, over $\mathbb{F}_p$.

e.g. $\dim V = 2$. We have $\{v_1, v_2\}$ – basis of V over $\mathbb{F}_p$.

Every $v \in V$ has the form
$v = \lambda_1 v_1 + \lambda_2 v_2$ ← will soon prove λ_1, λ_2 are unique

p options for each

$\lambda_1, \lambda_2 \in \mathbb{F}_p$.

$$\# V = p^2$$

$\# V = p^n$ ← n coefficients, p options for each one

$\dim V = n$

(as expected: think of $\mathbb{F}_p^n$ ← clearly has cardinality p^n).

Example The space of all solutions to

$f'' + f = 0$ (subspace of all smooth real fns) ← infinite-dimensional

- $\sin x$, $\cos x$ satisfy it.
- Any solution is $c_1 \sin x + c_2 \cos x$ $c_1, c_2 \in \mathbb{R}$

$\boxed{\text{2-dimensional}}$ (have not proved: there is no other solution lin. indep. from $\sin x$, $\cos x$)

Proofs:

easy lemmas:

- Call a set of linearly indep. vectors $\{v_1, \ldots, v_n\}$ maximal if you cannot add any vector to it and keep it linearly independent.

Lemma. A maximal linearly indep. set is a basis.

Pf: We want to prove that maximal $\Rightarrow$ spanning.

Suppose it is not spanning.

Then exists v s.t. $v \neq \lambda_1 v_1 + \cdots + \lambda_n v_n$ for any choice $\lambda_1, \ldots, \lambda_n$

$\vec{v}$

Then $\{v_1, \ldots, v_n, v\}$ is still lin. indep.

So $\{v_1, \ldots, v_n\}$ was not maximal.

Theorem: Suppose $\{v_1, \dots v_\ell\}$ is a lin. indep. set in V and $\{w_1, w_2, \dots, w_s\} \subset V$ s.t.

$$L(v_1, \dots v_\ell, w_1, \dots, w_s) = V.$$

Then there exists a subset of $\{w_1, \dots, w_s\}$ s.t. together with $v_1, \dots v_\ell$ it forms a basis.

Proof: intuition: $\{v_1, \dots v_\ell\}$ - lin. indep.

either it is spanning, then we are done: take $\emptyset \subset \{w_1, \dots w_s\}$

if not spanning, we know that once all the w's are thrown in, it becomes a spanning set.

but maybe it has become lin. dependent.

So maybe we should not have put in all w's.

Then we try to put the w's in one-by-one:

find the w_i s.t. $w_i \notin L(v_1, \dots v_\ell)$ (b/c all v's do not span but with the w's, they do.)

Put it in.

Because $w_i \notin L(v_1, \dots, v_\ell)$, the set $\{v_1, \dots, v_\ell, w_i\}$ is linearly independent.

Either $\{v_1, \dots v_\ell, w_i\}$ spans V, we are done

or it doesn't.

Then repeat the same argument.

In the book, this proof is done by induction on s: the number of w's.

(please read 3.4 - induction optional).