Vector spaces

Example (Main)

Think about $\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}$

$$= \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{R}\}.$$

We can do: addition of n-tuples

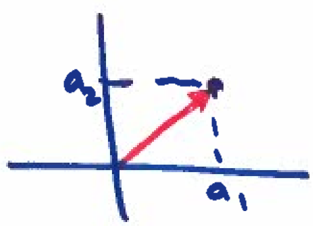
$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$

and: scalar multiplication:

$t \in \mathbb{R}$ $\qquad t \cdot (a_1, \ldots, a_n) = (ta_1, \ldots, ta_n)$

in pictures:

$\mathbb{R}^2$



can represent a tuple by a vector

(for now, a vector for us is just the tuple of its components.

$\bar{v} = \langle a_1, \ldots, a_n \rangle$

We will use $(a_1, \ldots, a_n)$ instead.

Some sources use $\hat{v}$, $\bar{v}$ to denote vectors.

We just say: let $x \in \mathbb{R}^n$ be a vector, $x = (x_1, \ldots, x_n)$

Note: for now we do not think of a vector as "magnitude + direction" - do not yet have the notion of either. Later, we'll define extra structure on a vector space, which will give us magnitudes and angles ("Euclidean spaces").

Want to move from this example to an abstract notion of a vector space.

<u>Def</u>: A <u>vector space</u> over $\mathbb{R}$ is a set V with two operations

"over the set of real numbers"
field

<u>addition</u>: for every $x, y \in V$, an element $x+y \in V$

<u>scalar multiplication</u>: for $t \in \mathbb{R}$, $v \in V$
$t \cdot v \in V$.

<u>satisfying the axioms</u>:

addition axioms

1) $(x+y)+z = x+(y+z)$ – associativity of addition
2) $x+y = y+x$ – commutativity of addition
3) There exists $\bar{0} \in V$ s.t. $\forall x \in V$ $x+\bar{0} = x$ (existence of 0)
4) $\forall x \in V$, $\exists (-x) \in V$ s.t. $x + (-x) = \bar{0}$
(existence of additive inverse)

5) $\forall t \in \mathbb{R}$, $s \in \mathbb{R}$, $x \in V$ $t \cdot (sx) = (ts) \cdot x$
6) $1 \cdot x = x$ for all $x \in V$
7) $\forall t \in \mathbb{R}$, $x, y \in V$ $t \cdot (x+y) = t \cdot x + t \cdot y$
8) $\forall t, s \in \mathbb{R}$, $x \in V$ $(t+s) \cdot x = t \cdot x + s \cdot x$

<u>linearity</u>

<u>Remark</u>: Our example $\mathbb{R}^n$ satisfies these axioms.
($\bar{0} = (0, \dots, 0)$).

<u>Easy consequences of the definition</u>

1. Axioms say: "exists $\bar{0} \in V$". Could there be more than one such element?

No: (proof by contradiction):

suppose we had two zeroes: $\bar{0}_1$ and $\bar{0}_2$.

Then: $\bar{0}_1 = \bar{0}_1 + \bar{0}_2 = \bar{0}_2 + \bar{0}_1 = \bar{0}_2$

because $\bar{0}_2$ is a zero.

b/c $\bar{0}_1$ is a zero.

So $\bar{0}_1 = \bar{0}_2$.

Similarly: for every x there is only one $(-x)$.

We will use the notation $x-y$ for $x+(-y)$.

Example: $t \in \mathbb{R}$, then $t \cdot \bar{0} = \bar{0}$

~~wrong way to prove it~~:

$\bar{0} = (0, \dots, 0)$

$t \cdot \bar{0} = (t \cdot 0, \dots, t \cdot 0)$

$= (0, \dots, 0) = \bar{0}$

Works for $\mathbb{R}^n$ which is just one example of a vector space.

Need to do: proof from the axioms.

Pf: Let $t \in \mathbb{R}$. Want to prove: $t \cdot \bar{0} = \bar{0}$.

checking that $t \cdot \bar{0}$ satisfies def'n of "zero"

Means: need to prove that for $\forall x \in V$, $x + t \cdot \bar{0} = x$

Two cases: $t = 0$ or $t \neq 0$.

Consider $t \neq 0$ case first.

def of $\bar{0}$

$$x + t \cdot \bar{0} = \underbrace{\left(t \cdot \tfrac{1}{t}\right)}_{\mathbb{R}} \cdot \bar{x} + t \cdot \bar{0} = t \cdot \left(\tfrac{1}{t} x + \bar{0} \right)$$

(using $1 \cdot \bar{x} = x$)

by axiom (7)

$$= t \cdot \tfrac{1}{t} x = x$$

So $t \cdot \bar{0}$ satisfies the def'n of "zero", then it equals $\bar{0}$

because we just proved that zero is unique.

Case 2: $t=0$.

Let's prove a more general statement then:

$0\cdot x = 0$ for all $x \in V$, not just for $\bar{0} \in V$.

homework

Example of a vector space (Important!)

Let V be the ~~space~~ set of all functions $f:\mathbb{R}\to\mathbb{R}$

Define addition: $(f+g)(x) = f(x) + g(x)$ – a function from $\mathbb{R}\to\mathbb{R}$

scalar multiplication: $\underbrace{(t\cdot f)}_{\text{new function}:\,\mathbb{R}\to\mathbb{R}}(x) = t\cdot f(x)$

Exer check that this satisfies the def. of a vector space.

(Note: the zero in this space is the function:

the constant zero function:

$z(x) = 0$ for all $x \in \mathbb{R}$

graph of $z(x)$.