Last time Finished self-adjoint linear operators. (over $\mathbb{R}$).
→ symmetric matrix.
(recall: eigenvalues are real, there is an orthonormal basis of eigenvectors, so the matrix is diagonalizable)

Today: applications: (quick sketch)
- discrete dynamical systems
- stochastic matrices, probability Markov chains

Idea: Many situations where you have "state" (a bunch of numbers) and apply linear operator, get a new "state".

Then do it many times.

Example: "state" could be - populations of rabbits and foxes.

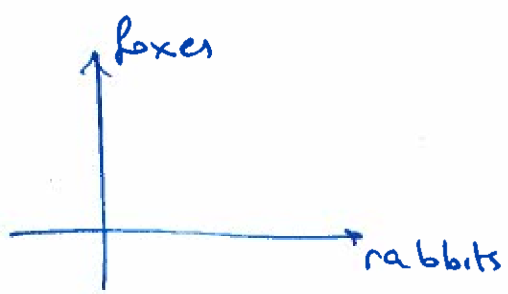
linear operator could be encoding how these populations influence each other:

- number of rabbits eaten by foxes is proportional to the number of foxes.

$$\begin{array}{ccc} & r & f \\ \begin{array}{c} r \\ \\ f \end{array} & \multicolumn{2}{c}{\begin{pmatrix} \overset{*}{a_{11}} & \overset{*}{a_{12}} \\ \overset{*}{a_{21}} & \overset{*}{a_{22}} \end{pmatrix}} \end{array}$$

born: proportional to how many you have

initial state: $\begin{pmatrix} R_0 \\ F_0 \end{pmatrix}$ ← # rabbits, ← # foxes



$$\begin{pmatrix} R_1 \\ F_1 \end{pmatrix} = A \cdot \begin{pmatrix} R_0 \\ F_0 \end{pmatrix}$$

populations after 1 year

2x2 matrix based on your predictions

$$R_1 = a_{11} R_0 + a_{12} F_0$$

neg. number accounting for fox food.

positive number "percentage of births" and death of rabbits.

After one year: $\begin{pmatrix} R_1 \\ F_1 \end{pmatrix} = A \begin{pmatrix} R_0 \\ F_0 \end{pmatrix}$

After n years: $\begin{pmatrix} R_n \\ F_n \end{pmatrix} = A^n \begin{pmatrix} R_0 \\ F_0 \end{pmatrix}$

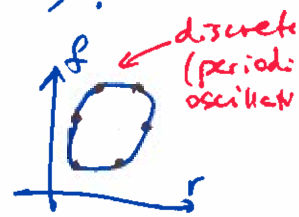
(Note: you could have seen predator-prey model in terms of differential equations:

$f(t)$, $r(t)$ — # foxes/rabbits at time t.

$$\underline{f'(t)} = \underline{a_{11}} f(t) + a_{12} r(t)$$

Our model is a discrete version of this).

(~~individual~~ individual states, not continuous time).

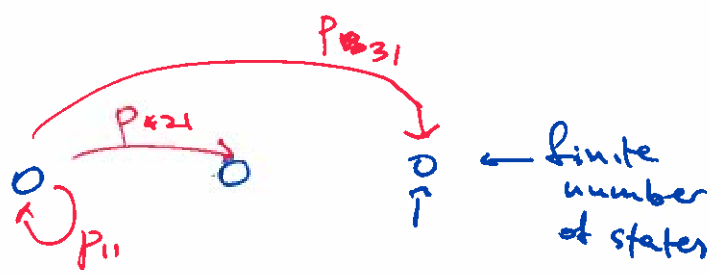


(Note: look at linear recurrences).

In many situations, such a system will approach a steady state: from any initial state, it will approach some fixed state as n gets large. ("equilibrium").

Finite automaton "vending machine"

Markov chain



Your imaginary machine is a list of probabilities p_{ji} of going from state i to state j

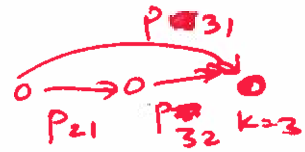
$p_{ij} \geq 0$. $\sum_{j=1}^{n} p_{ij} = 1$ ← we have to go somewhere from state i

Let $\bar{s} = (s_1, \dots, s_n)$ ← probabilities of our states.
(initially)

Let $P = (p_{ij})$

The new probabilities are $P\bar{s}$

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & & & \\ p_{n1} & - & \text{——} & p_{nn} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$



The k^{th} coordinate of this vector is the probability that we ended up in state k: we could have come to it from:

state 1 with prob. $p_{k1} \cdot s_1$

+

state 2 with prob. $p_{k2} \cdot s_2$

- ————

$$\sum_{j=1}^{n} p_{kj} s_j = \text{the } k^{th} \text{ entry of } P.s$$

The properties of P:
- $p_{ij} \geq 0$
- columns sum to 1.

positive stochastic matrix

(Note: predator-prey model doesn't satisfy these conditions).

Perron - Frobenius Theorem

1) A stochastic matrix has eigenvalue 1 and unique (up to scaling) eigenvector corr. to this eigenvalue

2) All other eigenvalues satisfy $|\lambda_0| < 1$.
(if $a_{ij} > 0$).

Pf: Let A be a stochastic matrix.
Consider A^T - the transpose of A.
Then the rows of A^T sum to 1.

$$A^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \text{sum of row } 1 \\ - \quad - \\ \text{sum of row } n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

← eigenvector of A^T with eigenvalue 1

But A and A^T have the same char. poly.

$$= \det(A - \lambda I) = \det(A^T - \lambda I^T) = \det(A - \lambda I)$$

↑ transpose doesn't affect det.

So 1 is an eigenvalue of A

So there must be an eigenvector of A.

$Aw = w$, Then $A^n w = w$. ← "steady state"

2) $A v = \lambda v$

"

$$\left(\sum_{j=1}^{n} a_{jk} v_j \right)_{k=1}^{n}$$

← easy to check.

$$\left| \sum a_{jk} v_j \right| \leq \sum |a_{jk}| |v_j| \leq v_j \underbrace{\left(\sum |a_{jk}| \right)}_{1} \Rightarrow |\lambda| < 1.$$

When A is diagonalizable, this means:

$$A^m = C^{-1} \cancel{\text{diag}} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^m C$$

$\lambda_1 = 1$.

$\to 0$ as $m \to \infty$ b/c $|\lambda_i| < 1$ for $i \neq 1$.

So: $A^m \longrightarrow$ projector onto the 1-dim subspace spanned by the steady state w.

Ex: Google page rank is "the steady state of the "google matrix".

(see the link on the webpage)