

Today: "Principal axes transformation"



- A self-adj. lin. op. in a Euclidean space has an orthonormal basis of eigenvectors

Prove it; talk about consequences and a recipe for finding this basis.

Last class: We already proved that if $A: V \rightarrow V$ is self-adjoint (ie. $\langle Av, w \rangle = \langle v, Aw \rangle$)

then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Corollary: If A has n distinct eigenvalues ($n = \dim V$) then A has a basis of eigenvectors that is orthonormal

(already orthogonal, normalize the eigenvectors to have length = 1).

Two issues: 1) Multiple eigenvalues: $\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}$ ← cannot happen!
2) Existence of real eigenvalues.

(here we are in a real Euclidean space!

What if all eigenvalues are in \mathbb{C} not in \mathbb{R} ?

↗
doesn't happen for self-adjoint lin. op. in V .

Our proof will go by induction on $\dim(V)$.

Base case: $V \cong \mathbb{R}$ $A: \mathbb{R} \rightarrow \mathbb{R}$ (self-adjoint.)

Then $Ax = \lambda x$
↑
eigenvalue (real by def.)

Nothing to prove.

normalize v to have length 1.



To prove the induction step, recall

orthogonal complement: if $v \in V$,

$$v^\perp = L(v)^\perp = \{ w \in V : w \perp v \}.$$

ie. $\langle w, v \rangle = 0$ (dim = $n-1$)

(Recall: $v^\perp \oplus L(v) = V$)

We need

Lemma $A: V \rightarrow V$ - a self-adj. lin. op.

Suppose v is an eigenvector for A .

Then v^\perp is an invariant subspace under A .

(More generally, if U is invariant for A , then U^\perp is also invariant)

Recall: Invariant subspace: a subsp. $U \subset V$ s.t. $A(U) \subset U$.

(Ex A 1-dim. subspace of V is invariant under A \Leftrightarrow it is spanned by an eigenvector of A).

Proof of the Lemma Need to prove: if $u \in v^\perp$ then $Au \in v^\perp$.

Meaning, want to check: $\langle v, Au \rangle = 0$.

$$\text{But } \langle v, Au \rangle \stackrel{\uparrow \text{self-adj}}{=} \langle Av, u \rangle \stackrel{\uparrow v \text{ is eigenvector}}{=} \langle \lambda v, u \rangle =$$

$$= \lambda \langle v, u \rangle \stackrel{\uparrow u \in v^\perp}{=} 0$$

So once we have an eigenvector v ,

we have $V = L(v) \oplus \underbrace{v^\perp}_{\substack{\text{dim} \\ n-1}} \leftarrow (n-1)\text{-dim space.}$

Because U is invariant, we can think of

$A|_U$ - "A restricted to U " - the lin. op. $A: U \rightarrow U$

So by induction assumption, $A|_U$ has an orthonormal basis of eigenvectors in U

The basis of V is $\left(\frac{v}{\|v\|}, \text{the basis of } U\right)$

This completes the proof, if we knew that at least one eigenvector exists.

Why is this a question? - how do we know that there is a real eigenvalue?

We need one more

Lemma $A: V \rightarrow V$ a self-adj. lin. op. \leftarrow a fact, all eigenvalues have to be real
Then it has a real eigenvalue $\delta, \omega \in \mathbb{R}$

Proof: We have a complex eigenvalue, $\lambda = \delta + i\omega$

A corresponding eigenvector is $x + iy$ (imagine that V becomes

$A: V \rightarrow V \leftarrow$ real space

λ - a root of $\det(A - \lambda I)$ - a poly with real coeffs.

$(\lambda \in \mathbb{C})$

Action of A on $x + iy$:

$$A(x + iy) = (\delta + i\omega)(x + iy)$$

means:

$$\begin{cases} Ax = \delta x - \omega y & \leftarrow \text{real part} \\ Ay = \delta y + \omega x & \leftarrow \text{imaginary part} \end{cases}$$

$V_{\mathbb{C}}$ - a complex vector space
 v_1, \dots, v_n - basis of V

look at

$$V_{\mathbb{C}} = \{z_1 v_1 + \dots + z_n v_n \mid z_i \in \mathbb{C}\}$$

So: $A(x + iy) = \overbrace{(\delta + i\omega)}^{\lambda} (x + iy)$

Still have $\langle Ax, y \rangle = \langle x, Ay \rangle$ ($x, y \in V$ - real vectors)

Plug in (x, y)

Get: $\langle \delta x - \omega y, y \rangle = \langle x, \delta y + \omega x \rangle$

$$\delta \langle x, y \rangle - \omega \langle y, y \rangle = \delta \langle x, y \rangle + \omega \langle x, x \rangle$$

$$\omega \cdot (\underbrace{\|x\|^2 + \|y\|^2}_{\neq 0}) = 0 \Rightarrow \boxed{\omega = 0}$$

so λ was real!

Consequences

Spectral Theorem If $A: V \rightarrow V$ is a self-adj. lin. op.

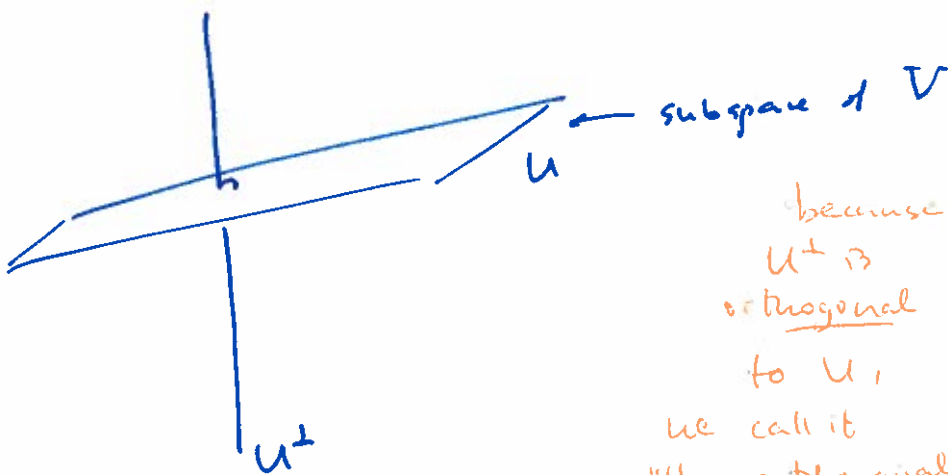
then we can write A as

$$A = \sum_{k=1}^m \lambda_k P_k$$

where $\lambda_1, \dots, \lambda_m$ are eigenvalues of A

and P_1, \dots, P_m are the ^{orthogonal} projectors onto the corresponding eigenspaces.

Explanation of the statement



because U^\perp is orthogonal to U , we call it "the orthogonal projector" onto U .

Recall: if we have

$$V = U \oplus U^\perp$$

can define the projector onto U along U^\perp .

$$P_U: V \rightarrow U$$

$$\text{Ker}(P_U) = U^\perp$$

$$P_U^2 = P_U$$

Spectral Thm says: $E_i = \text{Ker}(A - \lambda_i I)$ where λ_i are the eigenvalues of A

then $V = E_1 \oplus \dots \oplus E_m$. P_i - projector onto E_i

$$A = \sum_{i=1}^m \lambda_i P_i \Leftrightarrow A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$$

Important point What about the transition matrix?

Suppose $\{e_i\}$ is an ~~orthonormal~~ orthonormal basis in V .

What about the transition matrix that makes A diagonal?

! This matrix will be orthogonal (ie. will preserve the inner product

\Leftrightarrow that it takes an orthonormal basis to an orthonormal basis)

This is because we have an orthonormal basis in which A is diagonal.

Important point: Self-adjoint, what does it mean in terms of matrices:

Let $\{e_i\}$ be an orthonormal basis of V .

$A: V \rightarrow V$ is self-adjoint \Leftrightarrow its matrix with ~~resp.~~ resp. to $\{e_i\}$ is

\rightarrow Symmetric.

think + read.