Last time: We stated the Jordan Normal Form Thm:

$A : V \to V$ there exists a basis for V s.t. the matrix of A looks like this:

block-diagonal

each block is $\begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{pmatrix}$

(Note: $\boxed{\lambda} \leftarrow 1\times1$-block)

- Special case: $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ ← if V has a basis of eigenvectors

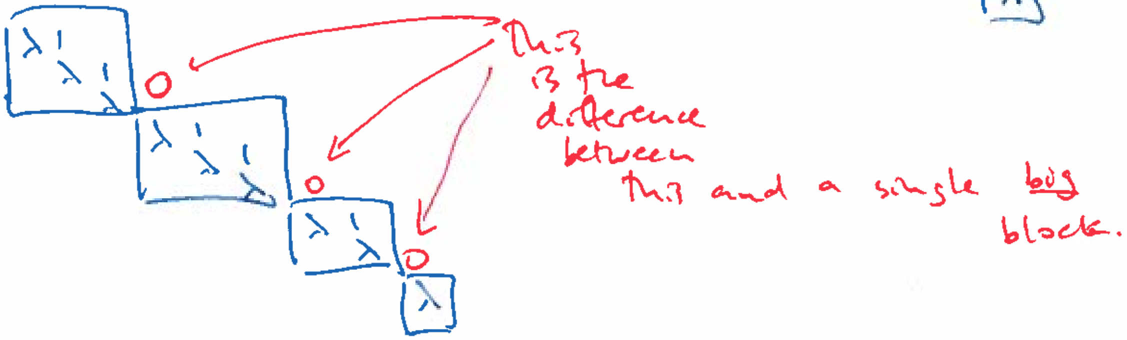
diagonal.

- Only multiple eigenvalues (roots of $p_A(\lambda) = \det(A - \lambda I)$ of multiplicity >1 give rise to blocks)

- We proved that if the eigenvalues are distinct, the matrix can be diagonalized.

- Warning: a repeated root of $p_A(\lambda)$ can give rise to several blocks: it can be:

or



We will not do it.

Note: in homework, we had a problem about a 2×2-block:

if λ is the only eigenvalue of $A: V \to V$, $\dim(V) = 2$,

then $\mathrm{Im}(A-\lambda I) \subset \ker(A-\lambda I)$

this was saying: for a 2-dim space, if λ has multiplicity 2, you get: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

i.e. $A = \lambda I$

or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$: $\ker(A-\lambda I)$ is 1-dim and coincides with $\mathrm{Im}(A-\lambda I)$

(i.e. $(A-\lambda I)^2 = 0$).

Reminder: change of basis $\longleftrightarrow$ $A^{new} = C^{-1} A^{old} C$

C = change of basis matrix.

Why do we care? "Matrix calculus"

- What if you wanted to apply the usual functions you know to matrices?
  - we know how to add and multiply matrices.

↑ note: $AB \neq BA$.
A^{-1} doesn't always exist if $A \neq 0$.

- Especially, one wants to compute powers of a matrix (which corresponds to repeatedly applying the same linear operator).

- Want to compute A^m. (m some high power)

Trick: get A to Jordan normal form:

$$B = C^{-1}AC$$

block matrix (normal form of A)

Then $B^m = \underbrace{(C^{-1}A\cancel{C})(\cancel{C^{-1}}AC)\dots(C^{-1}AC)}_{m}$

$$= C^{-1}A^mC$$

So $A^m = \boxed{CB^mC^{-1}}$

This is especially easy if B is diagonal:

$B = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ then $B^m = \begin{pmatrix}\lambda_1^m & & 0\\ & \ddots & \\ 0 & & \lambda_n^m\end{pmatrix}$

In particular, the eigenvalues of A^m are $\lambda_1^m \dots \lambda_n^m$!

Aside: with blocks: also not bad:

$$\underset{\substack{\uparrow\\ k\times k\text{-block}}}{\begin{pmatrix}\lambda & 1 & \\ & \ddots & 1\\ 0 & & \lambda\end{pmatrix}} = \underset{D}{\begin{pmatrix}\lambda & & 0\\ & \lambda & \\ 0 & & \lambda\end{pmatrix}} + \underset{N}{\begin{pmatrix}0 & 1 & 0\\ & 0 & 1\\ 0 & & 0\end{pmatrix}}$$

commute!

$DN = ND$.

$N^k = 0$ (N is nilpotent)

$(D+N)^m$ = write the binomial formula, many terms disappear because of high powers of N)

- What other functions can we apply?

  Exponential!

  $e^A = ??$

  what is e^x? (as a fn of x).

  - a function s.t. $(e^x)' = e^x$

    ie. solution to the equation $f' = f$.

    (really, get a family of solutions, $c \cdot e^x$), choose one s.t. $f(0)=1$ so $c=1$

    (e is defined as its value at 1)

  - To compute: Taylor series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

(recall $n! = 1 \cdot 2 \cdot 3 \cdots n$)

converges for all $x \in \mathbb{R}$

in fact, also converges for all $z \in \mathbb{C}$, defining e^z on $\mathbb{C}$

(!) Can use this to define

$$e^A = \underset{Id = "1".}{I_d} + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$$

If A is diagonal, $e^A = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

(with some work, prove all converges...)

- ! All other functions that have convergent Taylor series — can be applied to matrices.

For dif. equations: $f'(x) = \lambda f(x) \rightsquigarrow f(x) = c \cdot e^{\lambda x}$

Now you can deal with systems of dif. eq:

$f_1, \dots, f_n$

$$\bar{f}(x) = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}' = \underset{\uparrow}{A} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

matrix of scalars

constant vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Solution: $\bar{f}(x) = e^{Ax} \cdot C$

In practice, we diagonalize A, get $e^{\lambda_1 x} \dots e^{\lambda_n x}$

get solutions of the form $\sum c_i e^{\lambda_i x}$

/ Multiple eigenvalues lead to terms of the form $x e^{\lambda x}$, $x^2 e^{\lambda x}$ -- /

Also: higher order dif. eq. can be converted to systems of dif. eq: $f'' = \lambda f$

trick: make $f_1 = f$

$f_2 = f'$

$$f'' = \lambda f \text{ becomes } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}' = \begin{bmatrix} f_2 \\ \lambda f_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Self-adjoint linear operators

Def. Linear operators on Euclidean spaces (back to real vector spaces) that satisfy:

$$\langle f(v), w\rangle \overset{\uparrow}{=} \langle v, f(w)\rangle \quad \forall\, v, w \in V$$

inner product.

"self-adjointness" condition.

- Facts: 1) self-adj lin. operators are given by symmetric matrices w.r.t. an orthonormal basis.

  2) Any self-adjoint lin. op. has an orthonormal basis of eigenvectors (so it is diagonalizable)

Lemma: If v, w are eigenvectors of a self-adj. lin. op. $A: V \to V$ corresponding to distinct eigenvalues, then they are orthogonal.

Pf:

$$Av = \lambda v \qquad \lambda \neq \mu.$$
$$Aw = \mu w$$

compute $\langle Av, w\rangle = \langle \lambda v, w\rangle$

$$\boxed{\text{self-adj}} \longrightarrow \;\; \|$$

$$\langle v, Aw\rangle = \langle v, \mu w\rangle$$

So $\langle \lambda v, w\rangle = \langle v, \mu w\rangle$

$$\overset{\|}{\lambda\langle v, w\rangle} \qquad \overset{\|}{\mu\langle v, w\rangle}$$

So if $\lambda \neq \mu$, then $\langle v, w\rangle = 0$. ▨

(So if A has distinct eigenvalues, we are done with proof of (2)