Last time: ① change of basis in V of dim n over F

$\Leftrightarrow$ choosing an isomorphism (identification) $V \xrightarrow{\sim} F^n$

$v_1, \dots, v_n$ $\quad x = x_1 v_1 + \dots + x_n v_n \longmapsto (x_1, \dots, x_n) \in F^n$

② Changes of basis $\leftrightarrow$ transition matrices.
(every invertible matrix C can be thought of as transition matrix: the columns of C are the new basis.)

$$\boxed{X^{old} = C X^{new}}$$

← see corrected notes for the last class

③ For linear operators on V: $A: V \to V$ if A^{old} is a matrix for A in the old basis,

A^{new} = matrix for A in the new basis,

then $$\boxed{A^{new} = C^{-1} A^{old} C}$$

Def: Matrices A, B are called similar if there exists a matrix C s.t. $B = C^{-1} A C$

Two ways to think of it: 1) Fix a basis.
Then A, B, C are linear operators $: V \to V$.

In this case, A and B are called conjugate linear operators (they are NOT the same)

2) Similar matrices: exist choices of basis s.t. our similar matrices represent the same linear operator in one basis and the other, respectively.

Notation $GL_n(F)$ = "the group" of all invertible $n\times n$ - matrices with entries in F.
$\cap$
$M_n(F)$

Important aside • Being "similar" is an equivalence relation on $n \times n$-matrices

$\boxed{\S\, 11.1}$

Equivalence relation : A notion of "equivalence" for pairs of elements of some set :

Let S be a set. Some pairs of elements of S will be "distinguished" : we call them equivalent, and write $x \sim y$ (means, (x,y) is such a distinguished pair. Formally, an equiv. relation is a subset R of $S \times S$)

has to satisfy :

1) $x \sim x$ (i.e. $(x,x) \in R$ for all $x \in S$)
"reflexive"

2) $x \sim y \Rightarrow y \sim x$ (i.e., $(x,y) \in R \Rightarrow (y,x) \in R$)
("symmetric")

3) "transitive" : $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Examples: On the set of people :
- being of the same age is an equiv. relation
- being friends: No: not transitive.

In math: • For numbers
- being equal
- being congruent modulo given $d \in \mathbb{Z}$ is an equiv. relation on $\mathbb{Z}$ (denoted by $\equiv \bmod d$).

• For $n \times n$-matrices : being similar is an equiv. relation

$A, B \in M_n(F)$ $A \sim B \Leftrightarrow \exists\, C \in GL_n(F) : A = C^{-1}BC$

(note: we can rewrite this as $CA = BC$)

- Can make an equivalence relation on $n \times m$-matrices

say $A \sim B$ if there exists an $m \times n$-matrix C and an $m \times m$-matrix D s.t. both invertible

$$\underbrace{A C}_{\substack{(m\times n)\cdot(n\times m)\\ = m\times n\text{-matrix}}} = \underbrace{\underset{m\times m}{D}\,\underset{n\times m}{B}}_{n\times m\text{-matrix.}}$$

This means the following:

$B, A : V \to W$

$$\begin{tikzcd} \overset{n\text{-dim}}{V} \arrow[r, "A"] & \overset{m\text{-dim}}{W} \\ V \arrow[u, "C", "\wr"'] \arrow[r, "B"'] & W \arrow[u, "\wr", "D"'] \end{tikzcd}$$

"diagram commutes":

$DB : V \to W$

$AC : V \to W$

$$m\left[\begin{array}{c} \quad n \quad \end{array}\right]\left[\begin{array}{c} \quad n \quad \\ \\ C \end{array}\right] n$$

$= m \times n$

This means: we can change the basis in V and independently change the basis in W.

Rank Theorem Under this relation, any two $m \times n$-matrices of the same rank are equivalent!

Pf: Can choose bases in V and W so that the matrix looks like:

$$\left[\begin{array}{ccc|c} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

see hw: row operations $\Leftrightarrow$ changes of basis in V, W

Much harder theorem

Back to $n \times n$-matrices, $A : V \to V$

Back to similarity relation: you only choose a basis in V one time

$$A \sim B \iff \exists\, C \in GL_n(F) : \quad A = C^{-1} B C$$

How much can we simplify a matrix up to similarity?

- If the characteristic polynomial of A has n distinct roots

$$p_A(\lambda) = \det(A - \lambda I)$$

then A has a basis of eigenvectors

In this basis, A is diagonal!

(for each v_i, $Av_i = \lambda_i v_i$, so the ith column of A is $\begin{bmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{bmatrix} \leftarrow$ ith place)

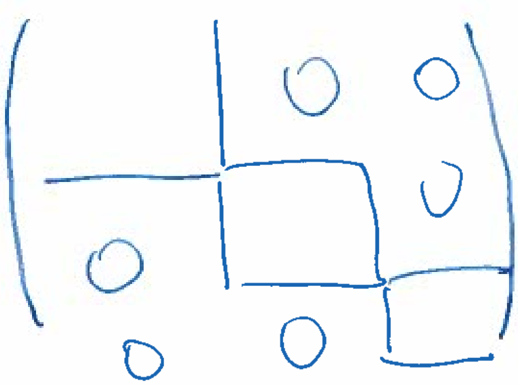
in the eigenvector basis, the matrix of A

is $\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$.

- If $p_A(\lambda)$ has multiple roots: $p_A(\lambda) = (x - \lambda_1)^{k_1} \cdots (x - \lambda_n)^{k_n}$

then cannot always make it diagonal:

But can get it to <u>Jordan Normal Form</u> (cannot prove)



each block looks like:

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}$$

the same eigenvalue

For a multiple eigenvalue, get smth like:

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 0 & \\ & & & \boxed{\lambda} & 0 \\ & & & & \boxed{\lambda} \end{pmatrix}$$

← size 3 block

← size 1 blocks

λ has algebraic multiplicity 5.

$P_A(\lambda)$ has root of mult. 5 at λ.

each block $\leftrightarrow$ one eigenvector.

here $\dim \ker (A-\lambda I)$ is 3.

Aside

Hard part: dealing with the situation when we do not have "enough" eigenvectors: when geometric multiplicity $\dim \ker (A-\lambda I)$ is smaller than algebraic multiplicity of λ.

not part of this course

Then we need to find more basis vectors $v_1, v_2, \ldots$ such that:

$$(A-\lambda I)v_1 = v_0 \quad \leftarrow \text{the eigenvector}$$

$$(A-\lambda I)v_2 = v_1$$

$$\ldots$$

this is hard because $A-\lambda I$ is not invertible! So hard to prove they exist; also they are not unique – how to make good choices?

This gives the Jordan block

$$\begin{array}{ccc} v_0 & v_1 & v_2 \end{array}$$
$$\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$$

here:

$Av_0 = \lambda v_0$

$Av_1 = \lambda v_1 + v_0$

$Av_2 = \lambda v_2 + v_1$