Today: • Equivalence of matrices
changes of basis

• State Jordan Normal form Theorem. (also rank Thm? - easy!)

Changes of basis

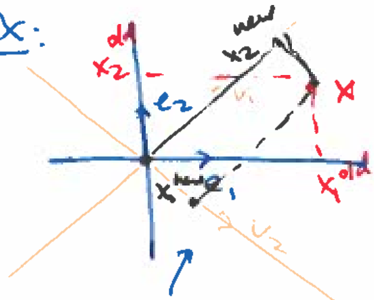
V - vector space over F.

$\{v_1^{old}, \dots, v_n^{old}\}$ - a basis of V

$\{v_1^{new}, \dots, v_n^{new}\}$ - a new basis of V.

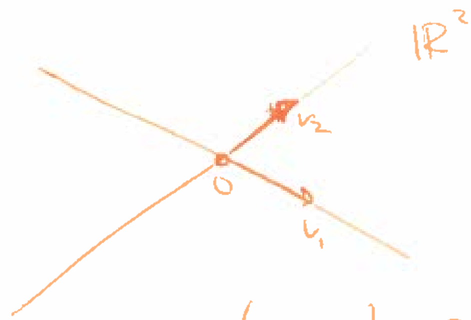
(Recall: they have the same number of elements, $n = \dim(V)$).

Ex: $\mathbb{R}^2$



"old" basis

$\mathbb{R}^2$



('new basis.

(often: old basis is $\{e_1, \dots, e_n\}$ - standard basis of F^n)

How do the old coords of x relate to the new coords of x?

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old} = x_1^{old} v_1^{old} + \dots + x_n^{old} v_n^{old}$$

Also

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new} = x_1^{new} v_1^{new} + \dots + x_n^{new} v_n^{new}$$

Geometrically hard to relate $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new}$ to $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old}$.

We solve this question algebraically:

we make a transition matrix C:

Columns of C are the coordinates of the new basis vectors in the old basis.

Example: Old basis: $\{e_1, e_2\}$ in $\mathbb{R}^2$

New basis: $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{old}$$

What are the new coordinates?

↑ usual standard coords.

We make C: $\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$ – our transition matrix

$\quad\quad\quad\quad\quad\quad\quad\uparrow v_1 \quad \uparrow v_2$

Aside: you could make a mean example:

$\{v_1 \ldots v_n\}$ – not the standard basis, given in terms of $e_1 \ldots e_n$

$\{w_1 \ldots w_n\}$ – another one. – given in terms of $e_1 \ldots e_n$.

Transition from $\{v_1 \ldots v_n\}$ to $\{w_1 \ldots w_n\}$?

Answer: C_1 = transition matrix from $\{e_1 \ldots, e_n\}$ to $\{v_1, \ldots, v_n\}$

C_2 – transition from $\{e_1, \ldots, e_n\}$ to $\{w_1, \ldots, w_n\}$

$$C_1 = \begin{bmatrix} \boxed{} & & \end{bmatrix} \qquad C_2 = \begin{bmatrix} & \boxed{} & \end{bmatrix}$$

coords of v_i w.r.t. $\{e_1, \ldots, e_n\}$

coords of w_i w.r.t. $\{e_1, \ldots, e_n\}$

To go from the v's to the w's:

$$\boxed{C^{-1} = C_2^{-1} C_1} \rightsquigarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new, v} = C_1^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{standard}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{w} = C_2^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{standard}$$

We have: $x^{old} = C \cdot x^{new}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C \cdot \begin{bmatrix} x_1^{new} \\ x_2^{new} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^{new} \\ x_2^{new} \end{bmatrix}$$

Then $$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{new} = C^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{old}}$$

why does C^{-1} exist?

we know: columns of C form a basis, so $rk(C) = n$

then C^{-1} exists (b/c $\dim(\ker(C)) = n - rk(C) = 0$.)

<u>Proof</u>:

$$\begin{tikzcd} V \arrow[d, "B^{new}"'] \arrow[dr, "B^{old}"] & \\ F^n \arrow[r, "C"'] & F^n \end{tikzcd}$$

choice of the basis $\{v_1 \ldots v_n\}^{new}$ means: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new} \in F^n$

$B : V \to F^n$ (old or new)

- the bijective linear operator that comes from a choice of a basis.

sends a vector to a tuple of its coords

C: columns are coords of the new basis w.r.t. the old basis:

it gives a map $: V \to V$ which takes old basis vectors to the new ones, but

what we are looking for is a map $: F^n \to F^n$ that agrees with it

(that would be the relation on the $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old}$ and $\begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix}^{new}$)

<u>have</u>: on the <u>new basis vectors</u> $v_i^{new} \xrightarrow{B^{old}} [v]$ ← column of its coords in the new basis and $B^{new}(v_i^{new}) = e$

Then $CB^{new}(v_i^{new}) = B^{old}(v_i^{new})$

$$\underset{B^{old}(x)}{\overset{\shortparallel}{x^{old}}} = C \underset{B^{new}(x)}{\overset{\shortparallel}{x^{new}}}$$

Then $B^{old}(x) = CB^{new}(x)$

For linear operators

$A : V \to V$ - lin. op.

has matrix A^{old} w.r.t. the old basis $\{v_1 \ldots v_n\}^{old}$

A^{new} — / — $\{v_1 — v_n\}^{new}$.

How are these matrices related?

$$x^{new} = C^{-1} x^{old}$$

$$\underbrace{\left(A^{old} x^{old}\right)}_{\in V}{}^{old} = \left(A^{new} x^{new}\right)^{new}$$

both are expressions for $\underline{Ax}$

lin. operator

Ax

$\left(A x^{old}\right)^{new}$ makes almost no sense

$x \in V \longrightarrow Ax \in V$

old coord, new coords; old, new coord.

Then:

$$C^{-1} A^{old} x^{old} = A^{new} x^{new}$$

$$C^{-1} A^{old} C x^{new} = A^{new} x^{new}$$

$$\boxed{A^{new} = C^{-1} A^{old} C}$$

Example: $C = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

recall: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ were eigenvectors for A

Compute $C^{-1} A C$: $\quad C^{-1} = \begin{bmatrix} -1 & -2 \\ -1 & 2 \end{bmatrix} \cdot \underbrace{\frac{1}{-4}}_{\text{"}\det A}$

using $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad = \begin{bmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix}$

$$C^{-1}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} C = \underbrace{\begin{bmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}}_{} \leftarrow A^{new}$$

$\boxed{\text{it does}} \longrightarrow$ expect $= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$