Today: • When does a basis of eigenvectors exist?
(a sufficient condition) 9.2

• How to go between different bases
(equivalence of matrices) → 11.1, 11.2, some 11.4

Recall: Last time we finished at the Fundam. Thm of Algebra:

every polynomial over $\mathbb{C}$ factors into a product of linear factors:

$$p(x) = a_n (x-\alpha_1) \cdots (x-\alpha_n) \quad (n = \deg(p)) \quad (\text{see } 9.4)$$

roots.

Comment: for $n \geq 5$, there is no formula for finding the roots! (Galois)

In practice, this means we cannot expect to find eigenvalues exactly. Numerically, can find eigenvalues approximately.

Def: If $p(x) = a(x-\alpha)^k \cdot h(x)$
the greatest such k is called the multiplicity of the root α.

• If start with an $n \times n$ matrix A,
write its characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$

$$\boxed{\begin{array}{l}\text{If } p_A(\lambda) \text{ has } n \text{ distinct roots in } \mathbb{C} \\ \quad (\text{meaning every root has multiplicity} = 1) \\ \text{then } A \text{ will have a basis of eigenvectors}\end{array}}$$

If there are multiple roots, it is more complicated.
- there might still be a basis of eigenvectors, or not. Not easy to distinguish these cases.

Recall why eigenvalues are roots of $p_A(\lambda)$:
because $\ker(A-\lambda I) \neq \{0\}$ when λ is an eigenvalue

Def: • $\dim \ker(A-\lambda I)$ is called the geometric multiplicit of the eigenvalue λ.

It is not 1 only if λ is also a multiple root of $p_A(\lambda)$ and in this case, the multiplicity of λ as a root of the char. poly. $p_A(\lambda)$ is called algebraic multiplicity.

Caution: geom. multiplicity $\leq$ alg. multiplicity
but they don't have to be equal!

To find geom. mult., you have to: do row reductions for $A-\lambda I$, find its kernel.

There is a method for finding a particularly good basis for this kernel, but it's not part of Math 223
And then need more basis vectors for V

Example: Recall from last time, the differentiation op.
$D: P_{\leq 2} \to P_{\leq 2}$, had matrix $\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ 0 & & 0 \end{bmatrix}$

it had eigenvalue 0 of algebraic multiplicity $=3$

what about geom. multiplicity:
$\ker(A-0\cdot I) = \ker(A)$ has dim 1, b/c
$\mathrm{rk}(A) = 2$.

Why geom. mult. $\leq$ alg. mult.
and why do we have a basis when the eigenvalues are distinct?

Thm If $\underbrace{v_1^{(1)}, \dots, v_{k_1}^{(1)}}$ are lin. indep. eigenvectors for eigenvalue $\underline{\lambda_1}$,

- -

$v_1^{(m)} \dots v_{k_m}^{(m)}$ are lin. indep. eigenvectors for the eigenvalue λ_m,

then $\{v_1^{(1)} \dots v_{k_m}^{(m)}\}$ are lin. indep.

all of them

(eigenvectors corr. to different eigenvalues are lin. indep.)

<u>based on</u>:

<u>Lemma</u>: If $\lambda_1 \dots \lambda_k$ are eigenvalues of $A: V \to V$, $\lambda_i \neq \lambda_j$ for $i \neq j$

Let v_i be an eigenvector for λ_i

then $v_1, \dots, v_k$ are lin. indep.

<u>Pf</u>: Induction on k.

<u>Base</u>: $\underline{k=1}$: v_1 - eigenvector for λ_1

$v_1 \neq 0$ by def'n.

Nothing to prove.

<u>Induction step</u>: Suppose we know this statement for $k=n$. Want to prove it for $k=n+1$.

Let $v_1 \dots v_{n+1}$ be eigenvectors for $\lambda_1, \dots, \lambda_{n+1}$.

Suppose we have

(1) $\boxed{\alpha_1 v_1 + \dots + \alpha_n v_n + \alpha_{n+1} v_{n+1} = 0}$ for some $\alpha_1, \dots, \alpha_{n+1} \in F$.

our field

Want to prove:

all $\alpha_i = 0$

Apply A: Get.

$$A(\alpha_1 v_1 + \dots + \alpha_{n+1} v_{n+1}) = 0.$$

$$= \alpha_1 A \cdot v_1 + \dots + \alpha_{n+1} A v_{n+1}$$

$$= \boxed{\alpha_1 \lambda_1 v_1 + \dots + \alpha_{n+1} \lambda_{n+1} v_{n+1} = 0.} \quad (2)$$

b/c eigenvectors.

Subtract $\boxed{\lambda_{n+1} \cdot (1)}$ from $\boxed{(2)}$:

Get: (the last term cancels!)

$$(\alpha_1 \lambda_1 v_1 - \alpha_1 \lambda_{n+1} v_1) + \cdots + (\alpha_n \lambda_n v_n - \lambda_{n+1} \alpha_n v_n) = 0$$

$$\Downarrow \quad \alpha_1 \underbrace{(\lambda_1 - \lambda_{n+1})}_{\neq 0} v_1 + \cdots + \alpha_n \underbrace{(\lambda_n - \lambda_{n+1})}_{\neq 0} v_n = 0.$$

By induction assumption, $\{v_i\}$ are lin. indep.

And it was given that $\lambda_i \neq \lambda_j$ for $i \neq j$

so $\lambda_i \neq \lambda_{n+1}$ when $i = 1, \ldots, n$.

Then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Thus we have $\alpha_{n+1} v_{n+1} = 0$. So $\alpha_{n+1} = 0$.

(b/c $v_{n+1} \neq 0$ by def. of an eigenvector)

Corollary: If the roots of the char. poly (over $\mathbb{C}$) are distinct, the lin. op. has a basis of eigenvectors

Pf: You have n eigenvalues, their eigenvectors are lin. independent.

$n = \dim(V)$

since there's n of them, they form a basis..

The matrix of A in this basis is diagonal!

How to change bases:

Changes of basis

- What happens to coordinates when you change the basis?
- What happens to matrices when you change the basis?

Let $v_1^{old}, \dots, v_n^{old}$ be a basis in V

And we take another basis, call them $v_1^{new} \dots v_n^{new}$.

$x \in V$ $\quad x = x_1^{old} v_1^{old} + \dots + x_n^{old} v_n^{old}$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{old} \text{ in the old basis.}$$

$$x = x_1^{new} v_1^{new} + \dots + x_n^{new} v_n^{new}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^{new} \text{ in the new basis.}$$

Make the "transition matrix" or "change of basis" matrix

$$C = \begin{bmatrix} c_{11} & & \\ \vdots & \cdots & \\ c_{1n} & & \end{bmatrix} \qquad v_i^{new} = \begin{bmatrix} c_{1i} \\ \vdots \\ c_{ni} \end{bmatrix} \leftarrow \text{coords in the } \underline{\text{old}} \text{ basis}$$

v_1^{new} columns are the coordinates of the $\underline{\text{new}}$ basis vectors w.r.t. the old basis.

we have: $\boxed{x^{old} = C x^{new}}$ ← easier to see

$$\boxed{x^{new} = C^{-1} x^{old}}$$

(proof next class)