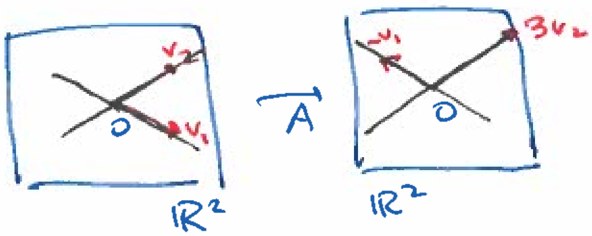
Recall: Example $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$: we figured out: $\lambda_1 = -1$, $\lambda_2 = 3$ - eigenvalues

with eigenvectors $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

determined up to a scalar

What are eigenvectors geometrically?

We had $A : \mathbb{R}^2 \to \mathbb{R}^2$



in the $\{v_1, v_2\}$ basis, A has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

eigenvalues on the diagonal

How do we know A? - we could try to say what it does "rotation, stretch... shear..."

or we give it by a matrix in the standard basis (this is what we did).

- Does A stabilize any line?

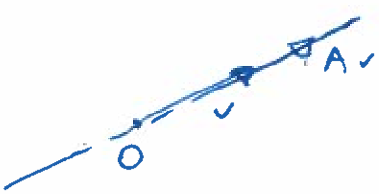
  the line doesn't move.

Any such line is a line spanned by an eigenvector:

stabilizing a line spanned by v means: $Av \parallel v$

$\Leftrightarrow Av = \lambda v$ for some scalar λ.

our def. of "eigenvector".



The point is: eigenvectors (if you can find them) are a more natural basis for our linear op.

Example Consider the space V of polynomials of degree ≤ 2
let $D: V \to V$ be the derivative: $p \mapsto p'$.
Find eigenvalues and eigenvectors.

Standard basis: $\{1, x, x^2\}$.
with resp. to this basis, the matrix for D is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = D$$

the first vector is an eigenvector with e-value 0.

The characteristic polynomial is $\det(D - \lambda I)$

$$= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)^3$$

Its roots: 0 of multiplicity 3

So, 0 is the only eigenvalue!

An eigenvector is: $D \cdot \underset{\substack{\uparrow \\ V}}{p} = 0 \Rightarrow p = \text{constant}$
$= c \cdot 1$

an eigenvect for $\lambda = 0$.

On the other hand, if $p' = 0$ then $p = \text{const.}$
So we have no other eigenvectors.

So this linear operator doesn't have a basis of eigenvectors!

Its only eigenvalue is 0 (of "algebraic multiplicity" 3

and we found one eigenvector

we proved that there are no other eigenvectors not parallel to our one (ours is $1 \leftrightarrow$ const. poly)

Example Consider the infinite-dimensional space $\underbrace{C^\infty(\mathbb{R})}_{V}$ of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ (they are infinitely differentiable, i.e. $f', f'', f''', \ldots$ all exist).

What are the eigenvectors of $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$
$$f \longmapsto f'$$

if λ is an eigenvalue, then
$$f'(x) = \lambda f(x) \iff f(x) = e^{\lambda x}$$
So every real number is an eigenvalue.

[**Aside:** Fourier transform is about decomposing $f \in V$ as a linear combination of the basis elements in this basis of eigenvectors.]

Some theory:

① • Why is $\det(A - \lambda I)$ a polynomial in λ? Does it always have roots?

② • When does A have a basis of eigenvectors?

③ • How to switch between bases?

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① $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = \underbrace{(a_{11} - \lambda)}_{\text{deg } n \text{ in } \lambda} \underbrace{\left| \quad \right|}_{\substack{\text{similar thing} \\ (n-1)\times(n-1)}} - a_{12} \underbrace{\begin{vmatrix} a_{21} & \cdots & a_{2n} \\ & \ddots & \\ & & \end{vmatrix}}_{\text{deg} \le n-2 \text{ by ind. assump.}}$$

deg $n-1$ poly in λ by ind. assumption.

one term contains λ in each row/column

Statement: If we have a determinant s.t. every row and column contains only one linear term in the variable λ, then such a $n\times n$ determinant is a polynomial of deg n in λ.

base: $n=1$: $|a_{11}-\lambda| \to$ deg 1 in λ.

Induction step: use cofactor expansion as above.
Get a sum of terms, one of them is degree n in λ, the others are lower degree.

Fundamental Theorem of Algebra (proved, e.g., using complex analysis)

The field of complex numbers $\mathbb{C}$ is algebraically closed, which means: every polynomial with coeffs in $\mathbb{C}$ has a root in $\mathbb{C}$.

Ex: in $\mathbb{R}$, this is not so: x^2+1 has no real root.

Magic: we made $\mathbb{C}$ by "putting in" a root of x^2+1.

Corollary: Over $\mathbb{C}$, every polynomial factors as a product of linear factors:

$$p(x) = \underset{\text{constant}}{a_n}(x-\alpha_1)(x-\alpha_2)\cdots\underset{\text{roots}}{(x-\alpha_n)}$$

$$p = a_n x^n + \cdots + a_0$$

A polynomial of degree n has exactly n roots in $\mathbb{C}$

Pf: a is a root of $p(x) \iff p(x) = (x-a)f(x)$ for some polynomial $f(x)$.

(Bezout's thm). So, use this + induction.

Def If $p(x) = (x-\alpha)^k f(x)$, k is called the multiplicity of α