Eigenvalues and eigenvectors

(often $F = \mathbb{C}$).

V - a vector space over a field F.

$A : V \to V$ - a linear operator (an endomorphism, meaning that it is from V to itself)

The point: there might exist a "convenient basis" for A.

(usually: we choose a basis in V, then write the matrix of A with resp. to that basis.

Instead, given a linear operator, can we find a basis ~~a~~ so that the matrix for that lin. op. is simple?)

ideally, diagonal: $\left(\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right)$

- Remark: The identity is one of the few nonzero lin. op. whose matrix does not depend on the choice of basis: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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other examples: projector: identity on U, 0 on a W - complement of U

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline 0 & & 0 \end{array}\right)$$

any basis that consists of a basis of U and a basis of W will give the same matrix

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One more series of examples: $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \lambda \, Id$

$\lambda \in F$

This scales every vector by λ.

Fact: The only lin. operators whose matrix does not depend on the choice of basis are of the form $\lambda \cdot Id$

(will prove later)

Suppose $A: V \to V$ - given lin. op.

Def: A vector $v \in V$ is called an eigenvector for A with eigenvalue $\lambda \in F$ if $Av = \lambda v$. $\boxed{v \neq 0}$

Will prove next class that they exist if $F = \mathbb{C}$.
Over arbitrary fields, ~~they~~ a given lin. op. might not have eigenvalues in that field.

Our goal: figure out how to find them when they exist.
And use them

Example: for $A = \lambda \cdot Id$, every vector in V is an eigenvector with eigenvalue λ.

Main observation: if $v \in V$ is an eigenvector for A with eigenvalue λ, then

$$v \in \ker(A - \lambda Id)$$

indeed, if $Av = \lambda v$, then $\underbrace{Av - \lambda v}_{= (A - \lambda \cdot Id) v} = 0$.

Then $\det(A - \lambda Id) = 0$
[write the matrix for $A - \lambda Id$ in any basis
(det does not depend on the choice of basis.)

Then the way to find eigenvalues is solve the equation $\det(A - \lambda Id) = 0$ for λ.

Example let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ ← mean, the lin. op. given by this matrix in the standard basis of $\mathbb{C}^2$.

Let's find its eigenvalues and eigenvectors.

The matrix of $A - \lambda Id$ (w.r.t. the standard basis)

is $\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}$

$$\underbrace{\det \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}}_{\text{quadratic polynomial in } \lambda} = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3.$$

We want: the values of λ for which it is 0!

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = -1 \text{ or } 3.$$

(This is why we are working over $\mathbb{C}$: over $\mathbb{R}$, we don't have to have roots!)

This one has real roots.

(The only way to solve these questions: find eigenvalues as roots of $\det(A - \lambda I)$, then solve for the eigenvectors)

How to find eigenvectors:

Now that we know the eigenvalues, we can make a system of equations:

for λ_1: $Av = \lambda_1 v$, or better: $(A - \lambda I)v = 0$.

$\lambda_1 = -1$: $\begin{bmatrix} 1-(-1) & 4 \\ 1 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

solve as usual: $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

it should have infinitely many solutions!

(if it doesn't, your "eigenvalue" is wrong).

$x_1 + 2x_2 = 0$, so $x_1 = -2x_2$.

Pick a vector that spans $\ker(A - \lambda \mathrm{Id})$.

Here, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ works.

(<u>Note</u>: if v is an eigenvector for A with eigenvalue λ, then $a \cdot v$ with $a \neq 0$ also satisfies this).

For $\underset{\substack{\shortparallel \\ 3}}{\lambda_2}$: $\begin{bmatrix} 1-3 & 4 \\ 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

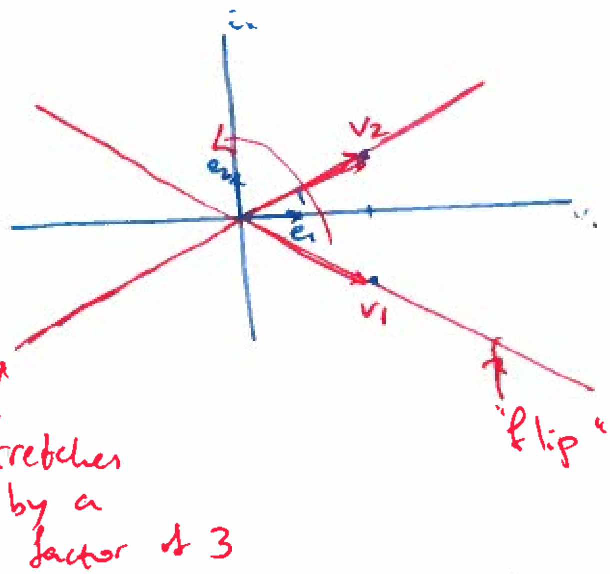
$$x_1 - 2x_2 = 0 \qquad x_1 = 2x_2$$

get: $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$.

So: for our A,

$\lambda_1 = -1$, eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix} = v_1$

$\lambda_2 = 3$, eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = v_2$



The matrix for A in the basis $\{v_1, v_2\}$ is $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ - diagonal matrix

eigenvalues on the diagonal.

$\boxed{\text{So our } A \text{ } \underline{\text{has}} \text{ an eigen}\underline{\text{basis}}}$

<u>Next time</u>: in general.