Last time Defined inner product $(,)$ on V ← a real vector space.

Makes it into Euclidean space.

↑ bilinear symmetric positive-def.

Example $(x,y) = x \cdot y$
$= \underbrace{x_1y_1 + \dots + x_ny_n}_{\text{dot product}}$

Today:
- Cauchy-Schwartz inequality
- Triangle inequality
- orthogonal complements, orthogonalization.

Recall: we can use an inner product to define the notions of length: $\|x\| = \sqrt{(x,x)}$

and angle: 

$\cos\alpha = \dfrac{(x,y)}{\|x\|\,\|y\|}$

Need to prove: $|(x,y)| \le \|x\|\,\|y\|$

(otherwise, $\cos^{-1}(\ldots)$ will not be defined).

Last time: proved that for our usual dot product, the notion of angle def'd this way agrees with our usual angle (law of cosines)

of course also the usual length: $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ is exactly the norm that comes from the dot product.

Today: Prove for any (x,y)-inner product.

Theorem $|(x,y)| \le \|x\|\,\|y\|$ ← Cauchy-Schwartz inequality

Pf: consider $(x-\lambda y, x-\lambda y) \geq 0$, where $\lambda = \frac{(x,y)}{\|y\|^2} \in \mathbb{R}$

$$\begin{aligned} &\| \\ &(x,x) - 2\lambda(x,y) + \lambda^2(y,y) \\ &= \|x\|^2 - \frac{2(x,y)}{\|y\|^2}\cdot(x,y) + \frac{(x,y)^2}{\|y\|^{\cancel{4}\,2}}\cdot\cancel{\|y\|^2} \\ &= \|x\|^2 - \frac{\cancel{2}(x,y)^2}{\|y\|^2} \geq 0 \end{aligned}$$

so $\boxed{\|x\|^2\|y\|^2 \geq (x,y)^2.}$

Generally, $(x,x) - 2\lambda(x,y) + \lambda^2(y,y)$

$$= \underbrace{\underset{c}{\|x\|^2} - 2\lambda\underset{b}{(x,y)} + \lambda^2\underset{a}{\|y\|^2}}_{\text{quadratic in } \lambda.} \geq 0 \quad \text{for all } \lambda.$$

/ Recall: $ax^2+bx+c \geq 0$ then $b^2-4ac \leq 0$. /

Then $4(x,y)^2 - 4\|x\|^2\|y\|^2 \leq 0$.

So, $(x,y)^2 \leq \|x\|^2\|y\|^2$

Example Consider the space of continuous functions on $[-1,1]$ (real-valued) with its inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,dx$$

(or: complex-valued functions, and define

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx \quad)$$

Then $(f,f) = \int_{-1}^{1} |f(x)|^2 dx \in \mathbb{R}$ $\quad (\bar{z} = a-bi,\ z = a+bi)$ check: $z\cdot\bar{z} = |z|^2$

we proved:

$$\int_{-1}^{1} f(x)\overline{g(x)}\,dx \le \sqrt{\int_{-1}^{1}|f(x)|^2dx \cdot \int_{-1}^{1}|g(x)|^2\,dx}$$

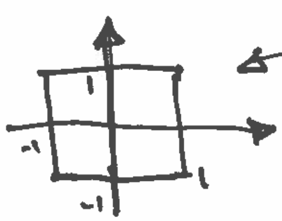
Properties of norms

- Recall $\|x\| = \sqrt{(x,x)}$ - exists b/c $(x,x) \geq 0$ by def'n.

The norm satisfies the properties:

1) $\|x\| = 0 \iff x = 0$, otherwise $\|x\| > 0$.
2) $\|\lambda x\| = |\lambda|\,\|x\|$
3) $\|x+y\| \le \|x\| + \|y\|$ - triangle inequality

aside: any function from V to $\mathbb{R}$ that satisfies these properties is called a norm on V.
Our norms are Euclidean norms (came from an inner product).
There are many other norms, example on $\mathbb{R}^2$:
let $|x| = \max(|x_1|, |x_2|)$
$\overset{\shortparallel}{(x_1,x_2)}$ what is the unit circle?
$\| \{x : \|x\| = 1\}$



- a square! ☺

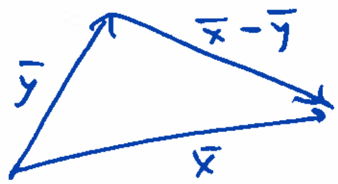
Proof of triangle inequality for Euclidean norms:

Consider $\|x+y\|^2 = (\cancel{x+y}, x+y) = (x,x) + 2(x,y) + (y,y)$

Cauchy-Schwartz $\longrightarrow$ $2(x,y) \le 2\|x\|\,\|y\|$

$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$.

Why is it called <u>triangle</u> inequality:



each side $\leq$ sum of lengths of the other two sides.

$$\|x\| \leq \|y\| + \|x-y\|$$

$\uparrow$ a $\quad$ $\uparrow$ b

$$\|a+b\| \leq \|a\| + \|b\|.$$

also,

$$\|x-y\| \leq \|x\| + \|y\|$$

$\uparrow$

"triangle inequality" says: $\|x-y\| \leq \|x\| + \|-y\|$

$= \|x\| + \|y\|$

<u>Orthogonal complements</u>

Let V be a Euclidean space.

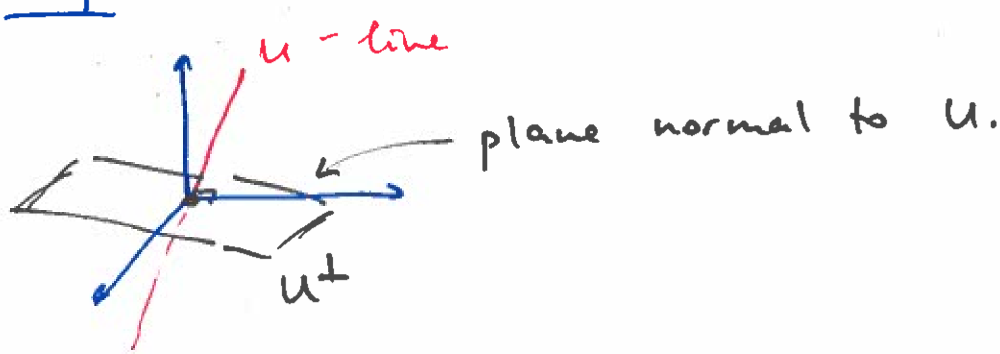
Let U be a subspace.

<u>Def</u>: $U^{\perp}$ "the orthogonal complement of U"

$= \{ v \in V : (v, u) = 0 \text{ for all } u \in U \}$

(the set of all vectors perpendicular to all vectors in U)

<u>Example</u>: $\mathbb{R}^3$, $(x, y) = x \cdot y$

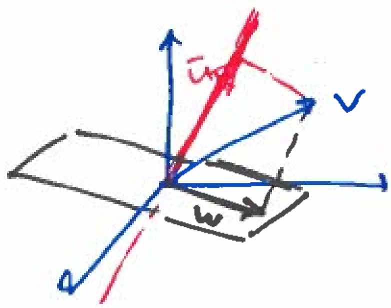


Theorem: Let U be a linear subspace of a Euclidean space V.

Then $V = U \oplus U^{\perp}$

(which means, each vector $v \in V$ has a unique decomposition $v = u + w$, where $u \in U$, $w \in U^{\perp}$)

Example:



$\mathbb{R}^3$

Next time: projections, proof of this.

Read: 8.1, 8.2, 8.3.