Today: Review

- Vector space (finite-dim.): **bases** (linear dependence / independence).

- Linear transformations.
  - over $\mathbb{R}$, $\mathbb{C}$ ex: $z \mapsto (1+3i)\cdot z$ in $\mathbb{C}$ as vector space over $\mathbb{R}$

    $f \mapsto f' + f''$ in the space of functions on $\mathbb{R}$.

    examples of "naturally" arising linear transf.

  - Need to choose bases in V, W

    $A : V \to W$ can be defined by a matrix.

    [columns are images of basis vectors]

- Row reductions $\rightsquigarrow$ gives you $\mathrm{rk}(A) = \dim(\mathrm{Im}(A))$
  - $\mathrm{Ker}(A) \longleftrightarrow$ systems of linear equations.
  - $\dim(\ker A) + \dim(\mathrm{Im}(A)) = \dim(V)$
  - A^{-1}, how it relates to solving equations
  - $\det(A)$.

Question

What linear transf. does A^t represent?

the answer is tricky (best seen with "orthogonal complements" will study starting next class)

Examples 1) Think of $\mathbb{C}$ as an $\mathbb{R}$-vector space.

Is $z \mapsto \frac{1}{z}$ a linear transf?

No!

a) $z \mapsto z^2+1$ No!

b) $\lambda \in \mathbb{R}$. $\lambda z \mapsto (\lambda z)^2 + 1 = \lambda^2 z^2 + 1 \neq \lambda(z^2+1)$

c) $z \mapsto \bar{z}$ ← complex conjugation

$\quad\;\; \shortparallel$

$a+bi \mapsto a-bi$

$a,b \in \mathbb{R}$

Is this linear ~~as~~ when we think of $\mathbb{C}$ as an $\mathbb{R}$-vector space

Yes:

$$\overline{(a_1+b_1 i) + (a_2+b_2 i)}$$

$$= \overline{(a_1+a_2) + (b_1+b_2)i}$$

$$= (a_1+a_2) - (b_1+b_2)i$$

$$= (a_1 - b_1 i) + (a_2 - b_2 i) = \bar{z}_1 + \bar{z}_2$$

Let $\lambda \in \mathbb{R}$. $\overline{\lambda(a+bi)} \underset{\lambda \in \mathbb{R}}{=} \overline{\lambda a + \lambda b i}$

$$= \cancel{\lambda(a+bi)}\ \underbrace{\lambda a}_{\in \mathbb{R}} - \underbrace{\lambda b}_{\in \mathbb{R}} i = \lambda \bar{z}.$$

Now think of $\mathbb{C}$ as a 1-dim vector space over $\mathbb{C}$.

Now is $z \mapsto \bar{z}$ a linear operator?

$\boxed{\text{No!}}$ We still have $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

But $\rightarrow$ $\lambda \in \mathbb{C} = \lambda + \lambda_2 i$

$$\overline{\lambda z} = \bar{\lambda}\,\bar{z} \leftarrow \text{exer}$$
$$\neq \lambda \bar{z}$$

Question: let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$

For what values of a do these NOT form a basis of $\mathbb{R}^3$?

Solutions: (partial answer: $a = \frac{1}{2}$ then $v_3 = \frac{1}{2} v_2$

so $\frac{1}{2}$ is one such value.

Others? How to figure it out?)

Solution 1: Form det.: $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & a \end{vmatrix}$ $\longrightarrow$ put them in as rows

$\{v_1, v_2, v_3\}$ is a basis $\Leftrightarrow$ they are lin indep.

b/c $\dim \mathbb{R}^3 = 3$

$\Leftrightarrow$ our det $\neq 0$.

$\Downarrow$

$a \begin{vmatrix} \_ & \_ \end{vmatrix} + \ldots = 0$

'linear eq. on a

will have one root. We already guessed it: $\frac{1}{2}$.

or instead of det,

Sol. 2 $\rightarrow$ you could just row reduce this matrix to find its rank.

- About equations and $\ker(A)$.

$$\begin{cases} x + 2y + 3z = 7 \\ -x + 2z = 0 \\ -x + \cancel{4}2y + \cancel{7}z = \cancel{14}7 \end{cases}$$

– find all solutions to this system.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ -1 & 0 & 2 & 0 \\ -1 & 2 & 7 & 7 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 2 & 5 & 7 \\ 0 & 4 & 10 & 14 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 2 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5/2 & 7/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

make this pivot = 1
$R_2 \to \frac{1}{2} R_2$

get to reduced echelon form

this column has no pivot ie. x_3 is a 'free' variable

$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 + \frac{5}{2} x_3 = \frac{7}{2} \end{cases}$$

Answer: $x_3 := t$ – a free parameter.

& $\begin{bmatrix} 2t \\ 7/2 - \frac{5}{2}t \\ t \end{bmatrix}$ – general solution in column form

Let A be the lin. op. with the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ -1 & 2 & 7 \end{bmatrix}$$ in the standard basis.

- Find ~~the~~ a basis of $\ker(A)$

Look at our solution:

$$\begin{pmatrix} 2t \\ 7/2 - 5/2\, t \\ t \end{pmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ 7/2 \\ 0 \end{bmatrix}}} + \underbrace{t \begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}}_{\text{in } \ker(A)}$$

the vector from RHS

(we were solving: $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix}$

Let $v^\circ = \begin{pmatrix} x_1^\circ \\ x_2^\circ \\ x_3^\circ \end{pmatrix} = \begin{bmatrix} 0 \\ 7/2 \\ 0 \end{bmatrix}$

We have $A v^\circ = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix}$

Every solution to $Ax = b$ has the form

$v^\circ + u$, where $u \in \ker(A)$

and $A v^\circ = b$

So the vector of coefficients of "t": $\begin{pmatrix} 2 \\ -5/2 \\ 1 \end{pmatrix}$

gives a basis for $\ker(A)$

More about ker(A):

To find ker(A), we would have to solve the equations $Ax = 0$. The system would look like this:

$$\begin{cases} x + 2y + 3z = 0 \\ -x \qquad + 2z = 0 \\ -x + 2y + 7z = 0 \end{cases}$$

Augmented matrix would be the same as above but with 0's on the right-hand side:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 2 & 7 & 0 \end{array}\right] \xrightarrow[\text{Row reductions}]{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row reductions same as above, the right-hand-side column is still 0

Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \ker(A) \iff \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + 5/2\, x_3 = 0 \end{cases}$

$$\iff \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -5/2\, x_3 \\ x_3 \end{pmatrix}$$

this vector is $\begin{pmatrix} 2 \\ -5/2 \\ 1 \end{pmatrix} \cdot x_3$

Thus, $\begin{pmatrix} 2 \\ -5/2 \\ 1 \end{pmatrix}$ is a basis vector for ker(A).