Today: <u>Inverting matrices</u>

- 6.4? "the <u>adjugate</u> matrix" - a ~~old~~ formula for A^{-1} that uses det
- 7.2 Cramer's rule - a formula for the solution to $Ax = b$ (system of linear equations) using det.

<u>Recall</u>: our algorithm for finding A^{-1} for a square matrix A.

- A^{-1} is a matrix such that $AA^{-1} = A^{-1}A = Id$

also the matrix of the inverse linear operator (so should exist if and only if $A : V \to W$ is an isomorphism. $\iff \begin{cases} \dim V = \dim W & (A \text{ is a square matrix}) \\ \ker(A) = \{0\} \end{cases}$

$\Updownarrow$ A is <u>injective</u>

$\iff \begin{cases} \dim(V) = \dim(W) \\ rk(A) = n \iff A \text{ is } \underline{\text{surjective}} \end{cases}$

Let A be $n \times n$-matrix.
To find A^{-1} we did:

$$\underbrace{\left(A \,\middle|\, \underbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{Id} \right)}_{\text{augmented matrix}} \xrightarrow[\substack{\text{elem. row operations} \\ \text{(No column operations)}}]{} \left(\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \,\middle|\, A^{-1} \right)$$

if didn't get a pivot in each column, then A^{-1} doesn't exist.

Reminder: this is equivalent to solving n systems of linear equations:

$$A x_i = e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ place} \quad \text{for } i=1, \dots, n$$

$x_i = A^{-1} e_i$ — becomes the i^{th} column of A^{-1}.

Today: Cramer's rule – formula for finding a solution for a system of linear equations $Ax = b$.

(A - a square matrix, assume A is invertible $\iff \det(A) \neq 0$) so we know the solution exists and is unique! $x = A^{-1} b$.

We have:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Imagine that $\begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}$ is the solution to this system of equations.

Take x_i^0 for one fixed value of i — the i^{th} component of the solution vector

Consider the matrix

$$M_i = \begin{pmatrix} a_{11} & \dots & (x_i^0 a_{1i} - b_1) & \dots & a_{1n} \\ \vdots & & & & \\ a_{n1} & \dots & (x_i^0 a_{ni} - b_n) & \dots & a_{nn} \end{pmatrix}$$

multiplied the i^{th} column of A by x_i^0 and subtract the RHS.

<u>Claim</u> : det of this matrix M_i is 0.
(because x_i^0 is the ith component of the solution.)

<u>Get</u>:

$$x_i^0 \det \begin{pmatrix} a_{11} - & a_{1n} \\ - & - \\ a_{n1} - & - a_{nn} \end{pmatrix} - \det \begin{pmatrix} a_{11} - & - & a_{1n} \\ a_{n1} & - & a_{nn} \end{pmatrix} = 0$$

ith column of A replaced by the vector $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

So

$$\boxed{x_i^0 = \frac{\det \begin{pmatrix} a_{11} \cdots & \quad & a_{1n} \\ a_{n1} - & \quad & a_{nn} \end{pmatrix}}{\det(A)}}$$

ith column is b

Cramer's rule.

<u>Pf of Claim</u>: our system of equations says:

$$a_{11} x_1^0 + a_{12} x_2^0 + - + a_{1n} x_n^0 = b_1$$
$$\vdots$$
$$a_{n1} x_1^0 + a_{n2} x_2^0 + - - + a_{nn} x_n^0 = b_n$$

x_i^0 appears in: $a_{1i} x_i^0$

$\vdots$

$a_{ni} x_i^0$

so we have: $M_i \begin{pmatrix} x_1^0 \\ -1 \\ x_n^0 \end{pmatrix} = 0$, i.e. $\ker(M_i) \neq \{0\}$.

in the ith place

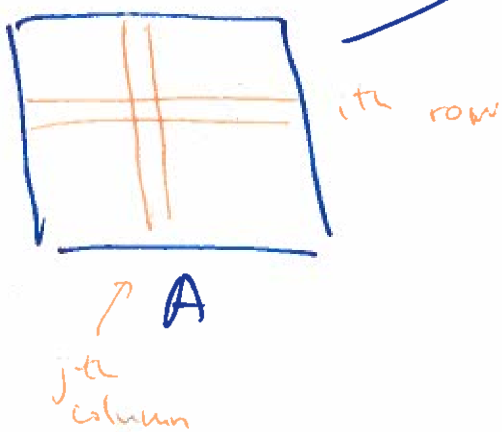
so $\det(M_i) = 0$.

A formula for A^{-1}

Assume A^{-1} exists $\Leftrightarrow \det(A) \neq 0$. ($A$ is an $n\times n$-matrix)

build the "adjugate" matrix of A.

Let $A_{ij} = \det\left(\quad \right)_{(n-1)\times(n-1)}$



throw away the i^{th} row and the j^{th} column.

Get an $(n-1)\times(n-1)$ matrix.

Let $$B = \left((-1)^{i+j} \det(A_{ij}) \right)^t$$

each entry is the det of a "minor" of A (the matrix A_{ij}) (up to sign)

Then $B = A^{-1} \cdot \det(A)$.

$$\boxed{A^{-1} = \frac{\left((-1)^{i+j}\det(A_{ij})\right)^t}{\det(A)}}$$

Example (Memorize the result!)

for 2×2-matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A_{ij} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

with signs: $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Now transpose: $\left((-1)^{i+j}A_{ij}\right)^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Then $\boxed{A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Check: $A^{-1}A = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \frac{1}{ad-bc}\begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\overset{\shortparallel}{0}$

Why does this formula work

The book: take $\left((-1)^{i+j}A_{ij}\right)^t \cdot A$

(in 6.4 ?)

see the diag. entries are $\det(A)$ by the formula for expansion of $\det(A)$ by the ith column.

off-diagonal entries are 0 because get matrices with 2 identical columns (requires thought.)

Our proof: use Cramer's rule

the ith column of A^{-1} is the vector of solutions to $Ax = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$th place.

use Cramer's rule: $x_j = \dfrac{\det \begin{array}{c} j\text{th column} \\ \begin{pmatrix} a_{11} & & 0 & & a_{1n} \\ & & 1 & \leftarrow i\text{th place} & \\ a_{n1} & & 0 & & a_{nn} \end{pmatrix} \end{array}}{\det A}$

$$= \frac{(-1)^{i+j} \det A_{ij}}{\det(A)}$$

row away jth column and ith row

expand the numerator in the ith column

a formula for the jth entry in the ith column of A^{-1}.

$$(A^{-1})_{ji} =$$

this is why you need to transpose

$$\text{So } A^{-1} = \frac{1}{\det(A)}\left((-1)^{i+j} \det(A_{ij}) \right)^t .$$

Monday: review. Please bring questions.
Review topics posted on the website under "announcements".