Today: * transpose matrices
* properties of det.

Friday: systems of equations (review)

Monday: a lot of review

Note: prepare for the midterm! (Fri: March 10)

Wednesday: 4.4 (quotient spaces) + review

After March 10: new things: Euclidean spaces, eigenvalues.

Last time stopped at "transposition": $A \rightsquigarrow A^t$

rows of A become columns of A^t.

Two important properties: ① $(A \cdot B)^t = B^t \cdot A^t$.

Note: Not $A^t B^t$!

(pf: try to write it / see the book)

② $\det(A) = \det(A^t)$. ← proof follows from the axioms and uniqueness.

More properties of det

#0) **Exer:** $\det\begin{pmatrix}1&2&3\\4&5&6\\2&4&6\end{pmatrix} = 0$ $\leftarrow$ Row 3 = 2× Row 1

recall: $\det \neq 0 \iff rk(A) = n \iff$ rows are lin. indep.
$\iff$ columns are lin. indep.

#1) $\begin{pmatrix}a_{11} & * & \cdots & * \\ 0 & a_{21} & \cdots & * \\ \vdots & 0 & \ddots & * \\ 0 & 0 & 0 & a_{nn}\end{pmatrix}$ $\leftarrow$ uppertriangular

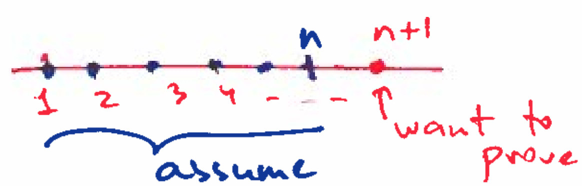
$\det(A) = a_{11}a_{21}\cdots a_{nn}$ – product of the diagonal entries

Pf: use induction

Aside: About induction, for statements of the form $P(n)$ ← depends on $n \in \mathbb{N}$.

(our case: $n \times n$ matrices).

method of proof: 1) prove it for some small n (usually, $n=1$)
(base case)



2) prove that $P(n) \Rightarrow P(n+1)$. "induction step"
Assume our statement is true for some $n \in \mathbb{N}$.
(or, for all $1, 2, 3, \ldots, n$)

- <u>Base case</u> $n=1$ $A=(a_{11})$ $\det(A)=a_{11}$, done

  [not needed, but as "sanity check", can also do: $n=2$:
  $A=\begin{bmatrix} a_{11} & b \\ 0 & a_{22} \end{bmatrix}$ $\det(A) = a_{11}a_{22} - 0 = a_{11}a_{22}$.
  Also true]

- <u>Induction step</u>: Assume we know the statement for $n\times n$-matrices.
  Let's prove it for $(n+1)\times(n+1)$-matrices.

Let $A = \begin{bmatrix} a_{11} & \cdots & & * \\ & \ddots & & \\ & & a_{nn} & \\ 0 & & & a_{n+1\,n+1} \end{bmatrix}$ be an $(n+1)\times(n+1)$-upper-triang. matrix.

use decomposition of det by the last row:

$$\det(A) = 0+0 \;--\; +0 + \underbrace{(-1)^{\overbrace{(n+1)+(n+1)}^{\text{even!}}}}_{=\,+} \underbrace{\det\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}}_{=\, a_{11}\cdots a_{nn} \text{ by the induction assumption}} \cdot a_{n+1\,n+1}$$

$\boxed{(-1)^{i+j}} \rightarrow \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

note: all diagonal entries are +: $i+i=2i$ - even!

$$= a_{11} \cdots a_{nn}\, a_{(n+1)(n+1)}$$

so we proved the formula for $n+1$.

3) (optional) "by the principle of mathematical induction" this completes the proof.

#2) <u>det (product)</u>: A, B - $n \times n$-matrices

$\det(AB) = \det(A)\det(B)$.

<u>Pf</u>: fix B, think of $\det(AB)$ as a function of A.

(in the book) it satisfies axioms of det, except for: the value at Id is $\det(B)$.

Then it has to be $\det(A)\det(B)$.

<u>Another proof</u> (same idea, spelled differently)

imagine a "unit cube" in V

 $\xrightarrow{\text{apply } B}$  $\xrightarrow{\text{apply } A}$ volume gets scaled by $\det(A)$.

volume 1

becomes a parallelepiped of volume $\det(B)$

volume gets scaled by $\det(AB)$

#3) <u>Block-diagonal matrices</u>:

(or even block upper-triangular)

$$\underbrace{C = \left(\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right)}$$

& $* = 0$, "block diagonal"

$\det C = \det(A) \cdot \det(B)$.

Pf: check the axioms.

or - think of volumes: for block-diagonal:

$$C = \begin{array}{c|c} & \begin{array}{ccc} v_1 \cdots v_k & v_{k+1} \cdots v_n \end{array} \\ \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \end{array}$$

this means, our basis is $v_1 \ldots v_k, v_{k+1} \ldots v_n$

and C takes $L(v_1, \ldots, v_k)$ to itself $\leftarrow$ $C(v_1)$ has 0 entries for $v_{k+1} \ldots v_n$ so it lies in $L(v_1, \ldots, v_k)$

and takes $L(v_{k+1}, \ldots, v_n)$ to itself.

Another way to say it: we have $V = U \oplus W$

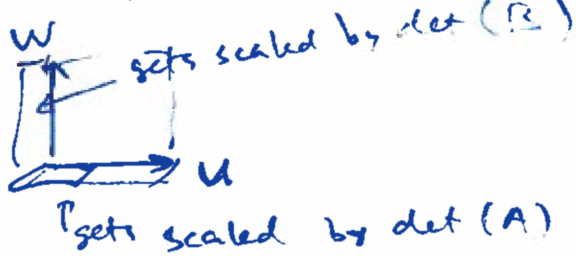
where $U = L(v_1, \ldots, v_k)$

$W = L(v_{k+1}, \ldots, v_n)$

and C takes U to U

W to W.

So what happens to volumes?



$\leftarrow$ volume scaled by $\det(A)\det(B)$.