Today:
- Finish Gaussian elimination.
- Talk about invertible matrices
- Determinants.

Example $A = \begin{pmatrix} 0 & 2 & 2 & 5 \\ 1 & -1 & 0 & 3 \\ 2 & 0 & 2 & 4 \end{pmatrix}$

- Find $\operatorname{rk} A$, a basis for $\ker(A)$

Solution: do Gaussian elim., get A into (reduced) row echelon form.

Row operations.

$$A \underset{R1 \leftrightarrow R2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 2 & 5 \\ 2 & 0 & 2 & 4 \end{pmatrix} \underset{R_3 - 2R_1}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 2 & 2 & 5 \\ 0 & 2 & 2 & -2 \end{pmatrix}$$

$$\underset{\frac{1}{2}R_2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & 5/2 \\ 0 & 2 & 2 & -2 \end{pmatrix} \underset{R_3 - 2R_2}{\rightsquigarrow} \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

pivots

$$\underset{\frac{1}{-7}R_3}{\rightsquigarrow} \underbrace{\begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{row echelon form}} \underset{R_1 + R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 & 11/2 \\ 0 & 1 & 1 & 5/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

pivot 1's

row echelon form (enough for finding rank: count the pivots)

now, go back kill all elements above the pivots

nothing to do

$$\underset{\substack{R_1 - \frac{11}{2}R_3 \\ R_2 - 5/2\, R_3}}{\rightsquigarrow} \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{reduced row echelon form}}$$

cannot improve it further w/o column operations.

Answering the questions: $\mathrm{rk}(A) = 3$
(3 pivots).

$A: \mathbb{R}^4 \to \mathbb{R}^3$, $\mathrm{rk}(A) = 3$
then $\dim(\ker(A)) = 1$.

Recall: $\dim(\ker(A)) + \mathrm{rk}(A) = \dim V = 4$

How to find the basis vector for $\ker(A)$?

After row reductions, we got:

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0 \end{cases} \leftarrow$$

our reduced matrix

$$B \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3rd column did not have a pivot

conditions for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \ker(A)$

Get: (*) $\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_4 = 0 \end{cases}$

free variable (any value of x_3 gives you a solution)

columns w/o pivots give you free variables

$$\boxed{\#\text{ free variables} = \dim(\ker(A))}$$

So our kernel has basis $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

(set $x_3 = 1$)
plug into (*)

Any ~~eleeleet~~ element of $\ker(A)$ is of the form $\underset{\sim}{x_3} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

rename it "t".

A Note about invertible matrices

When does a matrix A have an inverse?

1) it has to be square $(n \times n)$
(corresp. to a linear operator $: V \to W$ of the same dimension)

(very often, think of $A: V \to V$
$A: \mathbb{F}^n \to \mathbb{F}^n$)

2) And it has to be an isomorphism of vector spaces

Given $\dim V = \dim W$ $\iff$ $\dim(\ker(A)) = 0$: $\ker(A) = \{0\}$ (injective)

$\iff$ $\operatorname{rk}(A) = n$ (i.e., A is surjective)

3) Equivalently, row reduced echelon form of A is the Identity!
(when we do row reductions, need to get a pivot in every column, otherwise it means $\operatorname{rk}(A) < n$)

Example: (of a non-invertible matrix):

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{already reduced}}$$

another example: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ * & * & * & * \end{pmatrix}$

rank ≤ 3 depending on the $*$

Determinants

Warm-up: Find $\boxed{\dim \operatorname{Hom}(V, W) = m \cdot n}$

if $\dim V = n$

$\dim W = m$

choose a basis in V and W

linear maps: $V \to W$ $\longleftrightarrow$ $n \times m$-matrices

$\uparrow$

$n \cdot m$ entries

(need mn variables to determine such a linear map).

In particular, $\boxed{\begin{array}{l} \dim \operatorname{Hom}(V, V) = n^2 \\ \dim \operatorname{Hom}(V, F) = n \end{array}}$ $\dim(V) = n$.

Determinant (for only square matrices)

matrices

- Want a map D from $\operatorname{Hom}(V, V) \to F$

(1) which is linear in each of the rows of a matrix:

think of it as a function of $R_1, R_2, \ldots, R_n$

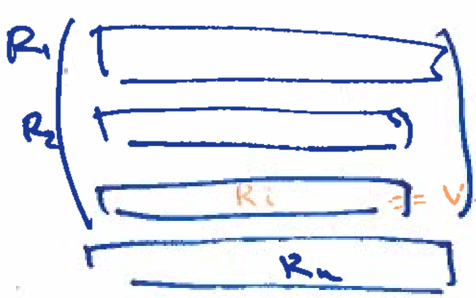
row vectors

s.t. if we fix all rows except R_i,

and change R_i to $R_i' = v + \lambda w$

then $D(R_1, \ldots, R_i', \ldots R_n) = D(R_1, \ldots, v, R_{i+1}, \ldots R_n)$

$+ \lambda D(R_1, \ldots, w, R_{i+1}, \ldots)$



$= v + \lambda w$

$v, w \in F^n$.

(2) • $D(A) = 0$ if $\mathrm{rk}(A) < n$

(i.e. $D(A) \neq 0 \iff A$ is invertible

(3) • $D(\mathrm{Id}) = 1$.

<u>Magic</u>: The space of such maps satisfying (1) and (2) is 1-dimensional

Property (3) makes this map <u>unique</u>.

For 2×2 - ~~b~~ matrices our map does this:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto ad - bc.$$

~~☒~~ <u>Exer</u>: check that it satisfies (1)–(3).

<u>Read</u> 6.1, 6.2.